Posterior Distribution of Non-differentiable Functions

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Motivation

- $\star$ This note was motivated by previous work on set-id SVARs.
  ($\lambda \in [\underline{g}(\theta)\ ,\ \overline{g}(\theta)]$, $\lambda \in \mathbb{R}$, $\theta \in \mathbb{R}^p$, $\underline{g}, \overline{g}$ nondifferentiable)

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- ⋆ 'Robust' Bayes credible sets require the posterior of $\underline{g}(\theta)$, $\overline{g}(\theta)$.
  (this approach also shows up in Kline & Tamer [16])
- ⋆ We wanted to understand the behavior of these posteriors
  (at least in situations where the sample size is large)

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  - ⋆ A Bernstein von-Mises Theorem holds for θ

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$\Longrightarrow$ Bayesians can use the bootstrap dist. of $g(\widehat{\theta}_n)$ as a posterior.
(May be simpler than MCMC; "poor man's posterior")

$\Longrightarrow$ Posterior credible sets for $g(\theta)$ need not be valid conf. sets.
(even if this relation holds true for θ)
(Dumbgen [93], Andrews [00] and Santos & Fang [16]).

Outline

1. Illustrative example: $g(\theta) = |\theta|$
2. Main Result: Asymptotic Behavior of Posterior
3. Corollary 1: Directional Differentiability
4. Corollary 2: Bootstrap fails $\implies$ Posterior fails
5. Conclusion

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- $\star$ Transformation: $g(\theta) = |\theta|$
- $\star$ There is a prior P over the parameter θ.

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- ⋆ Posterior draws for θ_n^{P*} can be expressed as:

$$\theta_n^j = \frac{Z_j^*}{\sqrt{n+\lambda^2}} + \frac{n}{n+\lambda^2}\widehat{\theta}_n, \quad Z_j^* \sim \mathcal{N}(0,1) \perp \text{prior,data.}$$

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- ⋆ Centered and scaled posterior draws of $g(\theta_n^j)$ are given by:

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Asymptotic equivalence

⋆ The centered and scaled posterior $\sqrt{n}(g(\theta_n^{P*}) - g(\widehat{\theta}_n))$:
(for prior $\theta \sim \mathcal{N}(0, \lambda^{-2})$)

$$\left(\left|\sqrt{\frac{n}{n+\lambda^2}} Z^* + \frac{n}{n+\lambda^2}\sqrt{n}\widehat{\theta}_n\right| - \left|\sqrt{n}\widehat{\theta}_n\right|\right), \quad Z^* \sim \mathcal{N}(0,1) \perp X^n$$

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- ⋆ The centered and scaled distribution of $\sqrt{n}(g(\theta_n^{B*}) - g(\widehat{\theta}_n))$:

$$\left(\left|Z^{**} + \sqrt{n}\widehat{\theta}_n\right| - \left|\sqrt{n}\widehat{\theta}_n\right|\right), \quad Z^{**} \sim \mathcal{N}(0,1) \perp X^n$$

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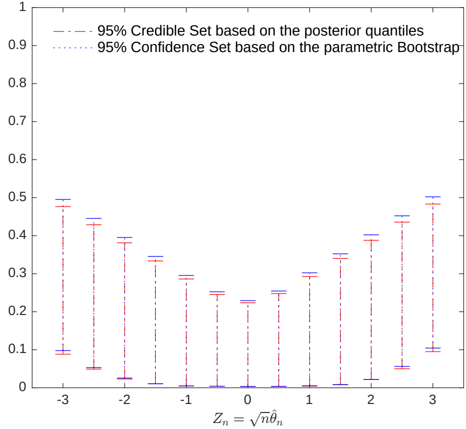
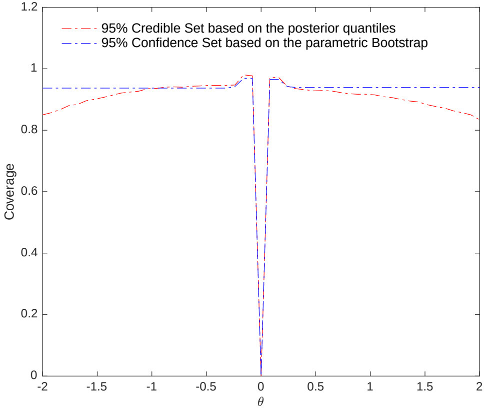
⋆ And we show that the 'distance' between these two r.v.'s $\to 0$.
(as the sample size goes to infinity)

Quantiles of $|\theta|$: $N(0, 1/5)$ Prior & Bootstrap



95% Credible set vs. 95% Bootstrap Confidence Set
($n = 100$, 10000 draws of Z^* and Z^{**})

Coverage of $|\theta|$: $N(0, 1/5)$ Prior & Bootstrap



95% Credible set and 95% Bootstrap Confidence Set
(n=100, 1000 data sets for each θ)

2. Main Result: Asymptotic Behavior of the Posterior of $g(\theta)$

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- The Bounded Lipschitz distance between the distributions induced by ϕ_n^* and ψ_n^* (conditional on the data X^n) is:

$$\beta(\phi_n^*, \psi_n^*;\ X^n) \equiv \sup_{f \in BL(1)} \Big| \mathbb{E}[f(\phi_n^*)|X^n] - \mathbb{E}[f(\psi_n^*)|X^n] \Big|.$$

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- ⋆ ϕ_n^* and ψ_n^* converge in Bounded Lipschitz distance in probability if:

$$\beta(\phi_n^*, \psi_n^*;\, X^n) \xrightarrow{p} 0, \qquad \text{as } n \to \infty$$

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Main Result:

If Assumptions 1 to 3 hold, then:

The Bounded Lipschitz distance between

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and

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goes in probability to zero.

PROOF (SKETCH):

- ⋆ Posterior and bootstrap distributions can be written as: (after centering and scaling)

$$\sqrt{n}(g(\theta_n^{P*}) - g(\hat{\theta}_n)) = \sqrt{n}(g(\theta_0 + Z_n^{P*}/\sqrt{n} + Z_n/\sqrt{n}) - g(\hat{\theta}_n)),$$
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Proof (Sketch):

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- We show that both distributions converge to a common limit:

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- To do so, we use the Lipschtz property of:

$$\Delta_n(a) \equiv \sqrt{n}(g(\theta_0 + a/\sqrt{n} + Z_n/\sqrt{n}) - g(\hat{\theta}_n))$$

3. Corollary 1: Directional Differentiability

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- ⋆ If Assumptions 1 to 4 hold, both random variables:

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and

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converge (in the BL metric in probability) to:

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Corollary 1

- ⋆ **A4:** g is Directionally Differentiable.
- ⋆ If Assumptions 1 to 4 hold, both random variables:

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- ⋆ Posterior of $g(\theta)$ need not be $\mathcal{N}$, even when posterior for θ is.

Proof (Sketch)

⋆ Because g is Lipschitz we have the approximation:

$$\begin{aligned}
&\sqrt{n}(g(\theta_n^*) - g(\widehat{\theta}_n)) \\
&\qquad = \sqrt{n}(g(\theta_0 + Z_n^*/\sqrt{n} + Z_n/\sqrt{n}) - g(\theta_0 + Z_n/\sqrt{n})) \\
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- ⋆ Same idea as in Santos & Fang [16].

4. Corollary 2: Credible sets for $g(\theta)$

Confidence and Credible sets

⋆ Let $q^*_\alpha(X^n)$ denote the quantile: (for $* \in \{B, P\}$)

$$q^*_\alpha(X^n) \equiv \inf_c \{c \in \mathbb{R} \mid \mathbb{P}^*(g(\theta^*_n) \leq c \mid X^n) \geq \alpha\}$$

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⋆ $CS^*_n(1-\alpha)$ fails to cover the parameter $g(\theta)$ at θ by at least $d_\alpha\%$ ($d_\alpha > 0$) if:

$$\limsup_{n\to\infty} \mathbb{P}_\theta\left(g(\theta) \in CS^*_n(1-\alpha)\right) \leq 1-\alpha-d_\alpha, \qquad (1)$$

where $\mathbb{P}_\theta$ refers to the distribution of X_i under parameter θ.

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⋆ Let $F_\theta(y|Z_n)$ denote the c.d.f. of the random variable

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⋆ **A5:** The c.d.f. $F_\theta(y|Z_n)$ is Lipschitz continuous with a constant k that does not depend on Z_n

Back to the Example

⋆ The p.d.f. of $Y \equiv g_0'(Z + Z_n) = |Z + Z_n|$ is:

$$h_0(y|Z_n) = \frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{1}{2}(y - Z_n)^2\Big) + \frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{1}{2}(y + Z_n)^2\Big)$$

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⋆ An analogous argument works for $y_1 < y_2$ and **A5.** holds

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- ⋆ Suppose that the nominal $(1-\alpha)\%$ bootstrap confidence set fails to cover $g(\theta)$ at θ by at least $d_\alpha\%$
- ⋆ Then, for any $0 < \epsilon < d_\alpha$,

$$\limsup_{n\to\infty} \mathbb{P}_\theta \left(g(\theta) \in CS_n^P(1-(\alpha+\epsilon)) \right) \leq 1-\alpha-d_\alpha$$

(i.e. the credible set:

$$CS_n^P(1-(\alpha+\epsilon))\% = \left[q^P_{(\alpha+\epsilon)/2}(X^n)\,,\; q^P_{1-(\alpha+\epsilon)/2}(X^n) \right]$$

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- ⋆ **I2:** Bootstrap c.set fails for $g(\theta) \implies$ Posterior fails as well

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- ⋆ Main result: Posterior distribution $\sim$ Bootstrap distribution
- ⋆ **I1:** Bayesians can use bootstrap for posterior inference
- ⋆ **I2:** Bootstrap c.set fails for $g(\theta) \implies$ Posterior fails as well
- ⋆ We hope this note gives a better idea of the limits of Bayesian inference when used for frequentist anaysis!

THANK YOU