Admissible, Similar Tests: A characterization. Appendix A. Proof of Theorem 1.

APPENDIX A: FINITE-SAMPLE THEORY

A.1. *Lemma 1: Weak* * *compactness of* $\mathcal{C}(\alpha\text{-}s)$

DESCRIPTION: The first lemma of this appendix shows that the set of α-similar tests, denoted $\mathcal{C}(\alpha\text{-s})$, is compact relative to the space of essentially bounded measurable functions endowed with the weak* topology.

RELEVANCE OF LEMMA 1: This lemma will be used to prove part ii) of Theorem 1. The weak* compactness of $\mathcal{C}(\alpha\text{-s})$ will allow the application of an essentially complete class Theorem [See Theorem 3, p. 87, Chapter 2 in Ferguson (1967)].

NOTATION: Let $\mathcal{B}(\mathbb{R}^n)$ denote the Borel σ-algebra on $\mathbb{R}^n$. For any set $\mathcal{S} \in \mathcal{B}(\mathbb{R}^n)$, let $\mathcal{B}(\mathbb{R}^n)_{\mathcal{S}}$ denote the sub-space σ-algebra. *Measurability* of the function $f : \mathcal{S} \to \mathbb{R}$ is always relative to the measurable spaces $(\mathcal{S}, \mathcal{B}(\mathbb{R}^n)_{\mathcal{S}})$-$(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. The integral of f with respect to the Lebesgue measure in $\mathbb{R}^n$ is denoted by $\int_{\mathcal{S}} f(s)ds$. *Integration* with respect to a different measure μ is denoted $\int_{\mathcal{S}} f(s)d\mu(s)$ or $\int_{\mathcal{S}} fd\mu$ if no ambiguity arises. All vectors are column vectors. For notational convenience, (a, b) will sometimes replace $(a', b')'$. The dimension of the column vector "a" is denoted d_a.

PRELIMINARIES 1 (L^1 and L^∞): Since the sample space $\mathbf{X} \in \mathcal{B}(\mathbb{R}^s)$, the triplet $(\mathbf{X}, \mathcal{B}(\mathbb{R}^s)_{\mathbf{X}}, \lambda^s)$ is a well-defined σ-finite measure space, where λ^S denotes the Lebesgue measure in $\mathbb{R}^s$ restricted to $\mathbf{X}$. Note that $\mathcal{B}(\mathbb{R}^s)_{\mathbf{X}} = \mathcal{B}(\mathbf{X})$ whenever $\mathbf{X}$ is endowed with the sub-space topology relative to $\mathbb{R}^s$. Following Rudin (2006), p. 65, let $L^1(\mathbf{X}, \mathcal{B}(\mathbf{X}), \lambda^s)$ denote the space of all real-valued measurable functions f that satisfy $||f||_1 \equiv \int_{\mathbf{X}} |f(x)|dx < \infty$. Let $L^\infty(\mathbf{X}, \mathcal{B}(\mathbf{X}), \lambda^s)$ denote the class of all essentially bounded real-valued measurable functions (Rudin (2006) p. 66).

REMARK 3: Identify the class of all tests $\mathcal{C}$ as a subset of $L^\infty(\mathbf{X}, \mathcal{B}(\mathbf{X}), \lambda^s)$

$$\mathcal{C} \equiv \{\phi \in L^\infty(\mathbf{X}, \mathcal{B}(\mathbf{X}), \lambda^s) \;\big|\; \phi(x) \in [0, 1] \text{ for } \lambda^s\text{-a.e. } x \in \mathbf{X}\}.$$

And note that the elements of any statistical model $\{f(x; \theta)\}_{\theta \in \mathbf{\Theta}}$ are elements of $L^1(\mathbf{X}, \mathcal{B}(\mathbf{X}), \lambda^s)$, by the definition of probability density function $\int_{\mathbf{X}} f(x; \theta)dx = 1 < \infty$ for all $\theta \in \mathbf{\Theta}$.

PRELIMINARIES 2 (The dual space of L^1): Let $[L^1(\mathbf{X}, \mathcal{B}(\mathbf{X}), \lambda^s)]^*$ denote the dual space of $L^1(\mathbf{X}, \mathcal{B}(\mathbf{X}), \lambda^s)$, i.e., the space of all continuous (w.r.t. $||f||_1$) linear functionals on $L^1(\mathbf{X}, \mathcal{B}(\mathbf{X}), \lambda^s)$; see Rudin (2005), p. 56. Let Λ denote an element of the dual space $[L^1(\mathbf{X}, \mathcal{B}(\mathbf{X}), \lambda^s)]^*$. By Theorem 6.16 in Rudin (2006), p. 127 and Theorem 1.18 in Rudin (2005), p. 15; the space $[L^1(\mathbf{X}, \mathcal{B}(\mathbf{X}), \lambda^s)]^*$ is isometrically isomorphic to $L^\infty(\mathbf{X}, \mathcal{B}(\mathbf{X}), \lambda^s)$. Therefore, one can identify each functional Λ with a unique element (up to equivalence) $g \in L^\infty(\mathbf{X}, \mathcal{B}(\mathbf{X}), \lambda^s)$, and vice versa: for $f \in L^1(\mathbf{X}, \mathcal{B}(\mathbf{X}), \lambda^s)^*$, the functional $\Lambda \in [L^1(\mathbf{X}, \mathcal{B}(\mathbf{X}), \lambda^s)]^*$ is of the form

$$\Lambda(f) \equiv \int_{\mathbf{X}} g(x)f(x)dx \quad \text{for some} \quad g \in L^\infty(\mathbf{X}, \mathcal{B}(\mathbf{X}), \lambda^s).$$

PRELIMINARIES 3 (weak* topology on L^∞): Endow the space $L^\infty(\mathbf{X}, \mathcal{B}(\mathbf{X}), \lambda^s)$ with the topology induced by the weak*-topology on the space $[L^1(\mathbf{X}, \mathcal{B}(\mathbf{X}), \lambda^s)]^*$; see Rudin (2005), p. 67, 68. The new topological space is denoted by $(L^\infty(\mathbf{X}, \mathcal{B}(\mathbf{X}), \lambda^s), \mathcal{T}^*)$. Denote convergence in such topology by $\to^*$. Note that, by definition, $\{g_n\}_{n\in\mathbb{N}} \to^* g$ if and only if

$$\int_{\mathbf{X}} f(x)g_n(x)dx \to \int_{\mathbf{X}} f(x)g(x)dx \quad \forall \quad f \in L^1(\mathbf{X}, \mathcal{B}(\mathbf{X}), \lambda^s).$$

Let $(\mathbf{X}, \mathbf{\Theta}, f, \Theta_0)$ be a hypothesis testing problem. Define

$$\mathcal{C}(\alpha\text{-}s) \equiv \left\{ \phi \in \mathcal{C} \;\middle|\; \mathbb{E}_\theta[\phi(X)] - \alpha] \equiv \int_{\mathbf{X}} (\phi(x) - \alpha) f(x;\theta) = 0 \quad \forall \theta \in \mathrm{Bd}\Theta_0 \quad \forall \right\}$$

Let $(L^\infty(\mathbf{X}, \mathcal{B}(\mathbf{X}), \lambda^s), \mathcal{T}^*)$ be the space of essentially bounded functions topologized with the weak* topology. For any $A \subset L^\infty(\mathbf{X}, \mathcal{B}(\mathbf{X})$, let $\mathcal{T}^*_A$ denote the subset topology induced by $\mathcal{T}^*$

Lemma 1: The set $\mathcal{C}(\alpha\text{-}s)$ is compact relative to $(\mathcal{C}, \mathcal{T}^*_{\mathcal{C}})$.

Proof: The outline of the proof is the following. I show that the set $\mathcal{C}(\alpha\text{-}s)$ is a sequentially closed subset of $\mathcal{C}$ with the relative weak* topology. Then I use the Banach-Alaoglu theorem and the topological separability of $L^1(\mathbf{X}, \mathcal{B}(\mathbf{X}), \lambda^s)$ to establish the compactness of $\mathcal{C}(\alpha\text{-}s)$.

(*Sequential Closedness*) Take any convergent sequence of tests $\phi_n \to^* \phi$ with $\{\phi_n\}_{n\in\mathbb{N}} \subseteq \mathcal{C}(\alpha\text{-}s)$. I want to show that $\phi \in \mathcal{C}(\alpha\text{-}s)$. First, I show that $\phi(x) \in \mathcal{C}$; i.e., $\phi \in [0,1]$ for almost every $x \in \mathbf{X}$. Suppose not. Then there exists a measurable set $A \in \mathcal{B}(\mathbf{X})$ with $\lambda^s(A) > 0$ such that $\phi(x) > 1$ or $\phi(x) < 0$ for all $x \in A$. Without loss of generality assume $\phi(x) > 1$. Since λ^s is σ-finite, there exists a countable collection $\{E_n\}_{n\in\mathbb{N}}$ such that $\cup_{n\in\mathbb{N}} E_n = \mathbf{X}$ and $\lambda^s(E_n) < \infty$ for every n. Consider the sequence of sets $\{A \cap E_n\}_{n\in\mathbb{N}}$. Note that $0 \le \lambda^s(A\cap E_n) < \infty$ for all $n \in \mathbb{N}$. In addition, there exists $N \in \mathbb{N}$ for which $0 < \lambda^s(A \cap E_N)$, otherwise $\lambda^s(A) = \lambda^s(\cup_{n=1}^\infty (A \cap E_n)) \le \sum_{n=1}^\infty \lambda^s(A \cap E_n) = 0$. Consider the indicator function $\mathbb{1}_{A\cap E_N}$. Since $0 < \lambda^s(A \cap E_N) < \infty$, the indicator function $\mathbb{1}_{A\cap E_N} \in L^1(\mathbf{X}, \mathcal{B}(\mathbf{X}), \lambda^s)$. Note that

$$\lambda^s(A \cap E_N) < \int_{\mathbf{X}} \mathbb{1}_{A\cap E_N}(x)\phi(x)dx = \lim_{n\to\infty} \int_{\mathbf{X}} \mathbb{1}_{A\cap E_N}(x)\phi_n(x)dx \le \lambda^s(A \cap E_N).$$

A contradiction. Therefore $\phi(x) \le 1$ λ^s-almost everywhere in $\mathbf{X}$. An analogous argument yields $\phi(x) \ge 0$ λ^s-almost everywhere. Therefore $\phi \in \mathcal{C}$. Now, I need to show that $\phi \in \mathcal{C}(\alpha\text{-}s)$. By assumption, for every $\theta \in \mathrm{Bd}\Theta_0$ $f(\cdot;\theta)$ is an element of $L^1(\mathbf{X}, \mathcal{B}(\mathbf{X}), \lambda^s)$. Consequently, $f(\cdot,\theta) \in L^1(\mathbf{X}, \mathcal{B}(\mathbf{X}), \lambda^s)$. Since $\phi_n \in \mathcal{C}(\alpha\text{-}s)$ for every $n \in \mathbb{N}$ weak* convergence yields

$$\begin{aligned} 0 = \lim_{n\to\infty} \int_{\mathbf{X}} f(x;\theta)(\phi_n(x) - \alpha)dx &= \Big(\lim_{n\to\infty} \int_{\mathbf{X}} f(x;\theta)\phi_n(x)dx\Big) - \int_{\mathbf{X}} f(x;\theta)\alpha dx \\ &= \int_{\mathbf{X}} f(x;\theta)\phi(x)dx - \int_{\mathbf{X}} f(x;\theta)\alpha dx \\ &= \int_{\mathbf{X}} f(x;\theta)(\phi(x) - \alpha)dx. \end{aligned}$$

So $\phi \in \mathcal{C}(\alpha\text{-}s)$. This implies $\mathcal{C}(\alpha\text{-}s)$ is sequentially closed in $\mathcal{C}$ endowed with the weak* topology.

(*Compactness*) Let

$$V \equiv \left\{ f \in L^1(\mathbf{X}, \mathcal{B}(\mathbf{X}), \lambda^s) \;:\; \int_{\mathbf{X}} |f(x)|dx \le 1 \right\}$$

Note that V is a neighborhood of the function $\mathbf{0}$ in the space $L^1(\mathbf{X}, \mathcal{B}(\mathbf{X}), \lambda^s)$. Let

$$K \equiv \left\{ g \in L^\infty(\mathbf{X}, \mathcal{B}(\mathbf{X}), \lambda^s) \;:\; \Big| \int_{\mathbf{X}} f(x)g(x) \Big| dx \le 1 \quad \forall \quad f \in V \right\}. \tag{A.1}$$

Note that $C(\alpha\text{-}s) \subseteq \mathcal{C} \subseteq K$, as for any test $\left| \int_{\mathbf{X}} f(x)\phi(x)dx \right| \le \int_{\mathbf{X}} |f(x)|\phi(x)dx \le \int_{\mathbf{X}} |f(x)|dx \le 1$. By the Banach-Alaouglu Theorem the set K is compact in the weak* topology; see Rudin

(2005), p. 68, Theorem 3.15. Furthermore, the space $L^1(\mathbf{X}, \mathcal{B}(\mathbf{X}), \lambda^s)$ is topologically separable as $(\mathbf{X}, \mathcal{B}(\mathbf{X}), \lambda^s)$ is a separable measure space; see exercise 10, Chapter 1 of Stein (2011). Therefore, Theorem 3.16 in Rudin (2005) p. 70 implies that the topological space $(K, \mathcal{T}_K^*)$ is compact and metrizable. Since every metrizable space is first-countable—consequently, Frechet-Urysohn— the sequential closure of C(α-s) coincides with its closure. Therefore, the set $C(\alpha - s)$ is a closed subset of the compact topological space $(K, \mathcal{T}_K^*)$. I conclude that $(\mathcal{C}(\alpha\text{-}s), \mathcal{T}^*_{\mathcal{C}(\alpha\text{-}s)})$ is compact and metrizable. That is, the space of α-similar tests is weak* compact. *Q.E.D.*

A.2. *Lemma 2: Tests that minimize risk in $\mathcal{C}(\alpha$-$s)$ are admissible in the class of all tests*

Description: Let $(\mathbf{X}, \Theta, f(x;\theta), \Theta_0)$ be a hypothesis testing problem with a product sample space $(\mathbf{X}_1, \mathbf{X}_2)$. $\Theta_0 \neq \emptyset$ is assumed to be a closed set relative to $(\Theta, \mathcal{T})$ and such that $\text{Bd}(\Theta_0) = \Theta_0$.

Relevance of Lemma 2: Lemma 2 will be used to establish part i) of Theorem 1.

Lemma 2: Let w_1 denote a full-support probability measures over int Θ_1. Define the *minimum average risk* over the set of α-similar procedures as:

$$M(w_1) \equiv \arg\min_{\phi \in \mathcal{C}(\alpha\text{-}s)} \int_{\text{int }\Theta_1} R(\phi, \theta) dw_1(\theta) \; ,$$

and suppose that Assumption F0 holds. Then,

L2a: If the sample space $\mathbf{X}$ is topologically separable:

$$M(w_1) \neq \emptyset.$$

L2b: Under Assumption F0,

$$\phi^* \in M(w_1) \implies \phi^* \text{ is admissible in } \mathcal{C}(\alpha\text{-}s).$$

L2c: Under Assumption F0:

$$\phi^* \in M(w_1) \implies \phi^* \text{ is admissible in } \mathcal{C}.$$

Proof of L2a: For simplicity, assume that w_1 has associated pdf p_1. I have shown that the class of tests $\mathcal{C}(\alpha$-s) is weak* compact. This class is non-empty, as it contains the randomized test $\phi(x) = \alpha$. To establish L2a it will be sufficient to show that the objective function

$$\mathcal{W}^*(\phi) \equiv \int_{\text{int }\Theta_1} R(\phi, \theta) p_1(\theta) d\theta$$

is continuous in the weak* topology.

L2a-Step 1 (Fubini's Theorem:) Since the image of any test $\phi \in \mathcal{C}$ is contained in the interval $[0,1]$ λ^s-a.e. and $f(x;\theta) \in L^1(\mathbf{X}, \mathcal{B}(\mathbf{X}), \lambda^s)$ for all θ, it follows that $\Big(\int_{\mathbf{X}} \phi(x) f(x;\theta) dx \Big) \leq 1$ for every $\theta \in \Theta$. Furthermore, since $p_1(x)$ is also a probability density functions on IntΘ_1 and IntΘ_0 the following holds

$$\int_{\text{Int}\Theta_1} \Big(\int_{\mathbf{X}} \phi(x) f(x;\theta) dx \Big) p_1(\theta) d\theta \leq 1 < \infty$$

Therefore, an application of Fubini's theorem in Billingsley (1995), p. 234 yields

$$\int_{\text{int }\Theta_1} R(\phi, \theta) p_1(\theta) d\theta \equiv \int_{\text{int }\Theta_1} \Big(\int_{\mathbf{X}} (1 - \phi(x)) f(x;\theta) dx \Big) p_1(\theta) d\theta = \int_{\mathbf{X}} (1 - \phi(x)) f_1^*(x) dx$$

where f_1^* is the "integrated" likelihood given by

$$f_1^*(x) \equiv \int_{\text{int }\Theta_1} f(x;\theta) p_1(x) d\theta, \tag{A.2}$$

Note that f_1^* is an element of $L^1(\mathbf{X}, \mathcal{B}(\mathbf{X}), \lambda^s)$. We can re-write

$$\mathcal{W}^*(\phi) \equiv \int_{\mathbf{X}} (1-\phi(x)) f_1^*(x) dx \tag{A.3}$$

L2a-Step 2 (Sequential Continuity of $\mathcal{W}^*$:) I now show that $\mathcal{W}^*$ is continuous on the compact metrizable space $(\mathcal{C}(\alpha\text{-}\mathcal{G}), \mathcal{T}^*_{\mathcal{C}(\alpha\text{-s})})$. It suffices to establish sequential continuity. Take any sequence of tests $\phi_n \to^* \phi$. Since f_1^* is an element of $L^1(\mathbf{X}, \mathcal{B}(\mathbf{X}), \lambda_{\mathbf{X}})$, convergence in the weak* topology yields

$$\int_{\mathbf{X}} \phi_n(x) f_1^*(x) dx \to \int_{\mathbf{X}} \phi(x) f_1^*(x) dx.$$

Therefore, the continuity of $\mathbf{W}$ implies

$$\begin{aligned} \mathcal{W}^*(\phi_n) &\equiv 1 - \int_{\mathbf{X}} \phi_n(x) f_1^*(x) dx \to 1 - \int_{\mathbf{X}} \phi(x) f_1^*(x) dx, \\ &= \mathcal{W}^*(\phi). \end{aligned}$$

Therefore, $\mathcal{W}^*$ is a continuous functional defined on the compact space $(\mathcal{C}(\alpha\text{-}s), \mathcal{T}^*_{\mathcal{C}(\alpha\text{-}s)})$, and $\mathcal{C}(\alpha\text{-}s) \neq \emptyset$, as it contains the test $\phi(x) = \alpha$. This implies $M(w_1) \neq \emptyset$.

L2b : Let $\phi^* \in M(w_1)$. I show that if $\phi' \in \mathcal{C}(\alpha\text{-}s)$ satisfies

$$\mathbb{E}_\theta[\phi'(X)] \geq \mathbb{E}_\theta[\phi^*(X)] \quad \forall \quad \theta \in \Theta_1 \tag{A.4}$$

then

$$\mathbb{E}_\theta[\phi'(x)] = \mathbb{E}_\theta[\phi^*(x)] \quad \forall \quad \theta \in \Theta_1. \tag{A.5}$$

Consequently, there is no test $\phi' \in \mathcal{C}(\alpha\text{-}s)$ that "weakly dominates" ϕ^* ; i.e, $R(\phi', \theta) \leq R(\phi^*, \theta)$ with strict inequality for some θ.

Suppose (A.4) hold, but (A.5) does not. Then, the following is true:

C1 There exists $\tilde{\theta} \in \Theta_1$ such that $\Delta_{\phi',\phi^*}(\tilde{\theta}) \equiv \mathbb{E}_{\tilde{\theta}}[\phi'(X)] - \mathbb{E}_{\tilde{\theta}}[\phi^*(X)] > 0$

C1 and the continuity of $\Delta_{\phi',\phi}(\cdot)$ at $\tilde{\theta}$ implies the existence of an open neighborhood $\tau_{\tilde{\theta}}$ for which $\Delta_{\phi',\phi^*}(\theta) > 0$ for all $\theta \in \tau_{\tilde{\theta}}$. Note that $\Theta_1 \neq \emptyset$ is an open set. It follows that the set $\mathcal{S}_{\tilde{\theta}}$ defined by $\mathcal{S}_{\tilde{\theta}} \equiv \tau_{\tilde{\theta}} \cap \Theta_1$ satisfies three properties: it is non-empty, it is open, and it is contained in Θ_1. Since $w_1(\theta)$ has full support on IntΘ_1, $\int_A dw_1(\theta) > 0$ for any open set A contained in Θ_1. Note that $\Delta_{\phi',\phi^*}(\theta) > 0$ for all $\theta \in S_{\tilde{\theta}}$ and (A.4) implies

$$\int_{\Theta_1} \Big(\int_{\mathbf{X}} (1-\phi'(x)) f(x;\theta) dx \Big) dw_1(\theta) < \int_{\Theta_1} \Big(\int_{\mathbf{X}} (1-\phi^*(x)) f(x;\theta) dx \Big) dw_1(\theta)$$

This contradicts the fact that $\phi^* \in M(w_1)$. I conclude C1 cannot hold.

Therefore, (A.4) implies (A.5). I conclude that ϕ^* is admissible in $\mathcal{C}(\alpha\text{-}s)$.

L2c : I now show that a test $\phi^* \in M(w_1)$ is admissible in the class of all tests. This proof is based on the arguments provided in Chernozhukov et al. (2009). The proof is divided into two steps.

Step 1: First I show that if $\phi' \in \mathcal{C}$ satisfies

$$\mathbb{E}_\theta[\phi'(X)] \leq \mathbb{E}_\theta[\phi^*(X)] \quad \forall \quad \theta \in \Theta_0 \tag{A.6}$$

and

$$\mathbb{E}_\theta[\phi'(X)] \geq \mathbb{E}_\theta[\phi^*(X)] \quad \forall \quad \theta \in \Theta_1 \tag{A.7}$$

with some strict inequality, then ϕ' is α-similar on $\text{Bd}\Theta_0 = \Theta_0$. Consequently, any test ϕ' that "weakly dominates" ϕ^* (i.e, $R(\phi', \theta) \leq R(\phi^*, \theta)$ with strict inequality for some θ) must be α-similar on the boundary of Θ_0.

Let $\mathcal{C}_{ns} \subset \mathcal{C}$ be the class of tests that are not similar on the boundary of Θ_0. This is, $\phi \in \mathcal{C}_{ns}$ if and only if there exists $\theta, \theta' \in \text{Bd}\Theta_0$ such that $\mathbb{E}_\theta[\phi(x)] \neq \mathbb{E}_{\theta'}[\phi(x)]$. Partition $\mathcal{C}$ according to $\mathcal{C}_{ns}$ so that $\mathcal{C} \equiv \mathcal{C}_{ns} \cup (\mathcal{C} \backslash \mathcal{C}_{ns})$. Take any test $\phi' \in \mathcal{C}_{ns}$ that satisfies (A.6). Since ϕ' is an element of $\mathcal{C}_{ns}$ and Θ_0 contains its boundary (as it is closed), there exists $\theta \in \text{Bd}\Theta_0$ such that $\Delta_{\phi',\phi^*}(\theta) \equiv \mathbb{E}_\theta[\phi'(X)] - \mathbb{E}_\theta[\phi^*(X)] < 0$. Because $\Delta_{\phi',\phi^*}(\theta) < 0$ and the function $\Delta_{\phi',\phi^*}(\cdot)$ is continuous at θ, there exists an open neighborhood $\tau_\theta \in \mathcal{T}$ such that $\Delta_{\phi',\phi^*}(\theta) < 0$ for all $\theta \in \tau_\theta$. Since θ is an element of $\text{Bd}\,\Theta_0$, then $\tau_\theta \cap \Theta_1 \neq \emptyset$. The latter implies there exists $\theta_1 \in \Theta_1$ such that $\Delta_{\phi',\phi^*_\alpha}(\theta_1) = \mathbb{E}_{\theta_1}[\phi'(X)] - \mathbb{E}_{\theta_1}[\phi^*_\alpha(X)] < 0$. Therefore, equation $(A.6)$ and $(A.7)$ cannot hold. We conclude there is no test $\phi' \in \mathcal{C}_{ns}$ that satisfies (A.6) and (A.7).
Since $\mathcal{C}_{ns}$ partitions $\mathcal{C}$, a test $\phi' \in \mathcal{C}$ that satisfies (A.6) and (A.4) must be an element of $\mathcal{C} \backslash \mathcal{C}_{ns}$ (as $\phi' \notin \mathcal{C}_{ns}$). Equation (A.6) implies ϕ' is α'-similar on the boundary with $\alpha' \leq \alpha$. Two cases follow: $\alpha' < \alpha$ or $\alpha' = \alpha$. In the first case, the argument in the previous paragraph implies that ϕ' will violate $(A.4)$. We conclude that any test that satisfies (A.6) and (A.4) must be α-similar on $\text{Bd}\Theta_0 = \Theta_0$.

Step 2: Since $\phi^* \in M(w_1)$, ϕ^* is admissible in $\mathcal{C}(\alpha\text{-}s)$. Therefore, there is no α-similar-on-the-boundary test such that $R(\phi', \theta) \leq R(\phi^*, \theta)$ with strict inequality for some $\theta \in \Theta$. Since —by Step 1— any test $\phi' \in C$ that that satisfies (A.6) and (A.4) must be α-similar on $\text{Bd}\Theta_0$, I conclude ϕ^* is admissible in $\mathcal{C}$

A.3. *Proof of Theorem 1*

Proof of Part i): The proof of the first part follows from Lemma 2. The argument goes as follows. Let $\mathcal{C}(\alpha - s)$ denote the class of α-similar tests. By definition

$$\begin{aligned}
\text{WAP}(\boldsymbol{\phi}^{w,\alpha}_{\text{WAP}}, w) &\equiv \int_{\Theta_1} \Big(\int_{\mathbf{X}} \boldsymbol{\phi}^{w,\alpha}_{\text{WAP}}(x) f(x;\theta)dx \Big) dw(\theta), \\
&= \int_{\Theta_1} \mathbb{E}_\theta \left[\boldsymbol{\phi}^{w,\alpha}_{\text{WAP}}(x)\right] dw(\theta), \\
&= 1 - \int_{\Theta_1} R(\boldsymbol{\phi}^{w,\alpha}_{\text{WAP}}, \theta) dw(\theta), \\
&\quad \text{(by the definition of risk function)} \\
&\geq \text{WAP}(\boldsymbol{\phi}, w), \quad \forall \boldsymbol{\phi} \in \mathcal{C}(\alpha - s).
\end{aligned}$$

This implies

$$\int_{\Theta_1} R(\boldsymbol{\phi}^{w,\alpha}_{\text{WAP}}, \theta) dw(\theta) \leq \int_{\Theta_1} R(\boldsymbol{\phi}, \theta) dw(\theta), \quad \forall \boldsymbol{\phi} \in \mathcal{C}(\alpha - s).$$

Implying WAP-similar tests of level α are average risk minimizers subject to an α-similarity constraint. Since Θ_0 is assumed open, then $\text{int}\Theta_1 = \Theta_1$. Lemma 2, part c, implies that $\boldsymbol{\phi}^{w,\alpha}_{\text{WAP}}$ is admissible in the class of all tests.

Q.E.D.

Proof of Part ii): The proof is based on the essentially complete class theorem; see Theorem 2.9.2 and 2.10.3 in Ferguson (1967) and also Le Cam (1986), Chapter 2, Theorem 1.

Note first that the class $\mathcal{C}(\alpha - s)$ is essentially complete relative to itself (as it contains all the α-similar tests). Note that the set $\mathcal{C}(\alpha - \text{s})$ is weak* compact by Lemma 1. In addition, the risk function of the testing problem $R(\phi, \theta)$ is—by definition of weak* topology—continuous (in ϕ) for all $\theta \in \Theta$. This verifies the assumptions of Theorem 2.9.2, p. 85, in Ferguson (1967).

Following Definition 3 Ferguson (1967) p. 50, $\phi^* \in \mathcal{C}(\alpha - s)$ is said to be an extended Bayes test if for every $\epsilon > 0$ there is a prior distribution $w_\epsilon(\theta)$ such that:

$$\int_{\Theta_1} R(\phi^*, \theta) dw_\epsilon(\theta) \leq \int_{\Theta_1} R(\boldsymbol{\phi}^{w_\epsilon,\alpha}_{\text{WAP}}, \theta) dw_\epsilon(\theta) + \epsilon.$$

Theorem 2.10.3 in Ferguson (1967) p. 87 implies that the set of *extended Bayes tests* in $\mathcal{C}(\alpha - \text{s})$ is *essentially complete*. This essential completeness means that for any other test $\phi \in \mathcal{C}$ there is a test ϕ^* extended Bayes in $\mathcal{C}(\alpha\text{-s})$ such that:

$$R(\phi^*, \theta) \leq R(\phi, \theta)$$

for all θ. Since ϕ is admissible and α-similar $R(\phi^*, \theta) \leq R(\phi, \theta)$ for all θ implies that $R(\phi^*, \theta) = R(\phi, \theta)$. Therefore, any admissible, α-similar test is risk equivalent to an extended Bayes test. This implies that for any $\epsilon > 0$ there is a probability measure w_ϵ such that

$$\text{WAP}(\phi, w_\epsilon) = \text{WAP}(\phi^*, w_\epsilon) \geq \text{WAP}(\boldsymbol{\phi}^{w_\epsilon,\alpha}_{\text{WAP}}, w_\epsilon) - \epsilon$$

Consequently, any admissible, α-similar test is an extended WAP-similar test of level α.

Q.E.D.

A.4. *Corollary to Theorem 1*

Corollary 1: *Suppose that Θ is compact. Let ϕ be an admissible, α-similar test. Let $\to^*$ denote convergence in the weak* topology as defined in Lemma 1 in Appendix A. Under Assumption F0 there exists a sequence of Borel probability measures w_n on Θ, a weight function w^*, and an α-similar test ϕ^* such that:*

$$w_n \overset{d}{\to} w^*, \quad \phi_{WAP}^{w_n,\alpha} \to^* \phi^*.$$

Moreover, if the sequence $\{w_n\}_{n\in\mathbb{N}}$ has a common σ-finite dominating measure, P, and the sequence of corresponding Radon-Nikodym derivaties $\{f_n\}_{n\in\mathbb{N}}$ admits a function g such that:

$$|f_n(\theta)| \leq g(\theta) \quad \textit{and} \quad \int_\Theta |g| dP < \infty,$$

then $WAP(\phi^*, w^*) = WAP(\phi, w^*)$. This means that for any admissible, α-similar test ϕ there is sequence of weights (with limit w^*) such that the test ϕ is WAP-equivalent to the properly defined limit of w_n-WAP α-similar tests.

Proof: I will break the proof of the corollary into 2 steps.

Step 1 (construction of w^* and ϕ^*): Take any sequence of real numbers $\{\epsilon_k\}_{k\in\mathbb{N}}$ such that $\epsilon_k \to 0$. Since ϕ is admissible and similar, Theorem 1 implies the existence of a sequence of Borel probability measures $\{w_k\}$ satisfying

$$\text{WAP}(\phi_{\text{WAP}}^{w_k,\alpha}, w_k) \geq \text{WAP}(\phi, w_k) \geq \text{WAP}(\phi_{\text{WAP}}^{w_k,\alpha}, w_k) - \epsilon_k. \tag{A.8}$$

The sequence $\{\text{WAP}(\phi_{\text{WAP}}^{w_k,\alpha}, w_k)\}_{k\in\mathbb{N}}$ takes its values on the $[0,1]$ interval. Hence, there exists a subsequence $\{w_{k_l}\}_{l\in\mathbb{N}}$ along which:

$$\text{WAP}(\phi_{\text{WAP}}^{w_{k_l},\alpha}, w_{k_l}) \to \text{WAP}^*,$$

where WAP^* is some number in the $[0,1]$ interval. Moreover, Equation (A.8) and $\epsilon_k \to 0$ then imply

$$\text{WAP}(\phi, w_{k_l}) \to \text{WAP}^*.$$

It is well known that if $\Theta \subseteq \mathbb{R}^p$ is endowed with its standard metric, then the space of Borel probability measures on Θ is sequentially compact—relative to the topology induced by the Prokhorov metric, which metrizes weak convergence (see Proposition 4.4 in Chapter F of Ok (2017)). Therefore, it is possible to extract a further subsequence $\{w_{k_{l_m}}\}_{m\in\mathbb{N}}$ such that:

$$w_{k_{l_m}} \overset{d}{\to} w^*.$$

In proving Theorem 1, I have showed that the space of α-similar tests is compact relative to the weak* topology, and also metrizable. The latter implies that the space of α-similar tests is sequentially compact. Consequently, I can extract a further subsequence $\{w_{k_{l_{m_n}}}\}_{n\in\mathbb{N}}$ along which:

$$\phi_{\text{WAP}}^{w_{k_{l_{m_n}}},\alpha} \to^* \phi^*,$$

as n tends to infinity. Consider thus the sequence of weights $\{w_n\}_{n\in\mathbb{N}}$ with n-th element defined as

$$w_n \equiv w_{k_{l_{m_n}}}.$$

By construction, under this sequence

$$w_n \overset{d}{\to} w^*, \quad \phi_{\text{WAP}}^{w_n,\alpha} \to^* \phi^*, \quad \text{WAP}(\phi_{\text{WAP}}^{w_n,\alpha}, w_n) \to \text{WAP}^*, \quad \text{WAP}(\phi, w_n) \to \text{WAP}^*.$$

with ϕ^* α-similar.

Step 2 (ϕ and ϕ^* are WAP-equivalent under w^*): I will now show that $\text{WAP}(\phi, w^*) = \text{WAP}^* = \text{WAP}(\phi^*, w^*)$. So that the original admissible, similar test ϕ and the limiting test ϕ^* are WAP equivalent under the limiting weight w^*.

First, Assumption F0 and the definition of weak convergence of probability measures implies

$$\text{WAP}(\phi, w_n) \equiv \int_\Theta R(\phi;\theta)dw_n \to \int_\Theta R(\phi;\theta)dw \equiv \text{WAP}(\phi, w^*).$$

(as $R(\phi, \cdot)$ is bounded and, by Assumption F0, continuous). Since—by construction of w_n—$\text{WAP}(\phi, w_n) \to \text{WAP}^*$, then $\text{WAP}(\phi, w^*) = \text{WAP}^*$.

Second, $\text{WAP}^* = \text{WAP}(\phi^*, w^*)$. To establish such a relation, it is sufficient to show

$$\text{WAP}(\phi^{w_n,\alpha}_{\text{WAP}}, w_n) \to \text{WAP}(\phi^*, w^*),$$

as, by construction, $\text{WAP}(\phi^{w_n,\alpha}_{\text{WAP}}, w_n) \to \text{WAP}^*$. Note that:

$$\begin{aligned}
\text{WAP}(\phi^{w_n,\alpha}_{\text{WAP}}, w_n) - \text{WAP}(\phi^*, w^*) &= \text{WAP}(\phi^{w_n,\alpha}_{\text{WAP}}, w_n) - \text{WAP}(\phi^*, w_n) \\
&+ \text{WAP}(\phi^*, w_n) - \text{WAP}(\phi^*, w^*) \\
&= \text{WAP}(\phi^{w_n,\alpha}_{\text{WAP}}, w_n) - \text{WAP}(\phi^*, w_n) + o(1) \\
&\quad \text{(By Assumption F0 and the definition of weak convergence)} \\
&= \int_\Theta (R(\phi^{w_n,\alpha}_{\text{WAP}};\theta) - R(\phi^*,\theta))dw_n + o(1) \\
&= \int_\Theta (R(\phi^{w_n,\alpha}_{\text{WAP}};\theta) - R(\phi^*,\theta))f_n(\theta)dP + o(1) \\
&\quad \text{(by the assumption about the existence of Radon-Nikodym} \\
&\quad \text{derivatives w.r.t. } P \text{ and equation (32.5) in Billingsley (1995))} \\
&= o(1)
\end{aligned}$$

To establish the last equality, define the sequence of functions:

$$h_n(\theta) \equiv (R(\phi^{w_n,\alpha}_{\text{WAP}};\theta) - R(\phi^*,\theta))f_n(\theta)$$

and note that $|h_n(\theta)| \leq |(R(\phi^{w_n,\alpha}_{\text{WAP}};\theta) - R(\phi^*,\theta))|g(\theta)$. Since $\phi^{w_n,\alpha}_{\text{WAP}} \to^* \phi^*$ it then follows that $R(\phi^{w_n,\alpha}_{\text{WAP}};\theta) \to R(\phi^*,\theta))$ for every θ and consequently:

$$h_n(\theta) \to 0 \quad \forall\, \theta \in \Theta.$$

An application of the dominated convergence theorem yields

$$\text{WAP}(\phi^{w_n,\alpha}_{\text{WAP}}, w_n) - \text{WAP}(\phi^*, w^*) \to 0.$$

Consequently,

$$\text{WAP}(\phi^*, w^*) = \text{WAP}^*.$$

Therefore ϕ and ϕ^* are WAP-equivalent under w^*. That is, $\text{WAP}(\phi, w^*) = \text{WAP}^* = \text{WAP}(\phi^*, w^*)$.

Q.E.D.

Comment: The essentially complete class theorem (ECCT) invoked in the proof of Theorem 1—based on Theorems 2.9.2 and 2.10.3 in Ferguson (1967)—characterizes admissible tests as extended Bayes tests. There are other versions of the ECCT that characterize admissible tests in terms of ‘limits of Bayes procedures’. For example, Theorem 4A.10 in Brown (1986) shows that the closure (in weak* topology) of the set of Bayes procedures for priors concentrated on finite subsets of Θ constitutes—under some assumptions on the action space, the loss function, and the statistical model—an essentially complete class. Note that if were able to verify such a theorem in our set-up, then for every admissible, α-similar test ϕ there would be a test ϕ^*—on the closure of Bayes procedures—for which $R(\phi^*, \theta) = R(\phi, \theta)$ for every θ. This, by definition of closure, would imply the existence of a sequence of weights w_n (concentrated on finite subsets of Θ) such that:

$$\phi^{w_n,\alpha}_{WAP} \to^* \phi^*,$$

and consequently, by the definition of weak* convergence,

$$R(\phi^{w_n,\alpha}_{WAP}, \theta) \to R(\phi^*, \theta) = R(\phi, \theta), \quad \forall\, \theta \in \Theta.$$

This is a stronger result than the one obtained in Corollary 1. To the best of my knowledge, the stronger version of the complete class theorems seem to require the convexity of the action space as well as strict convexity of the loss function (see for example Theorem 7.15 in Lehmann and Casella (1998).

A.5. *Lemma 3:* WAP-*similar tests with a boundedly complete, null-sufficient statistic.*

Preliminaries: This section generalizes a well-known observation in the IV literature: maximizing constrained average power is straightforward whenever there is a boundedly-complete, null-sufficient statistic. Consider the following assumptions.

Assumption F1 (Null Sufficiency): There is a partition of the data $X = (x_1, x_2)$ such that the conditional density of x_1 given x_2 in the statistical model $f(x_1, x_2; \theta)$ satisfies:

$$f(x_1|x_2;\beta_0) \equiv f(x_1|x_2;\theta) = f(x_1|x_2;\theta') \quad \forall\, \theta, \theta' \in \Theta_0. \tag{A.9}$$

The statistic x_2 arising from such partition of the data will be called a *null-sufficient statistic*.

It is well known that a null-sufficient statistic can be used to control the *null* rejection probability of a test in a two-sided problem with a nuisance parameter [Ferguson (1967), Moreira (2003), Andrews et al. (2006), Lehmann and Romano (2005)].

Let $h(x_2; \theta)$ denote the marginal density of the null-sufficient statistic x_2 based on the statistical model $f(x_1, x_2; \beta, \pi)$.

Assumption F2: (Bounded Completeness): For any bounded measurable function $m : \mathbf{X}_2 \to \mathbb{R}$, the marginal densities of the null sufficient statistic are such that:

$$\int m(x_2)h(x_2;\theta)dx_2 = 0, \quad \forall\, \theta \in \Theta_0 \implies m(x_2) = 0,$$

except, perhaps, in a set that has zero measure under every element of the collection $\{h(\cdot, \theta)\}_{\theta \in \Theta_0}$.[11]

Theorem 4.3.1 in Lehmann and Romano (2005) provides a sufficient condition to guarantee that a family of distributions is complete, and thus, boundedly complete. In the IV example studied in this paper, it will be sufficient to show that the set Θ_0 contains a rectangle of the same dimension as the null-sufficient statistic.

Bounded completeness will be used to show that all similar tests must be "conditionally" similar. This is a well-known result in the theory of statistical hypothesis testing. See Theorem 4.3.2 in Lehmann and Romano (2005).

Description: Lemma 3 will show that under assumptions F1, F2 the test that rejects whenever

$$\phi^*(x_1, x_2) \equiv f^*_{w_1}(x_1, x_2)/f(x_1|x_2;\beta_0) > c(x_2;\alpha), \tag{A.10}$$

is an element of

$$M(w_1) \equiv \underset{\phi \in \mathcal{C}(\alpha\text{-}s)}{\arg\min} \int_{\text{Int}\,\Theta_1} R(\phi, \theta) dw_1(\theta)$$

provided $c(x_2; \alpha)$ is the $1 - \alpha$ quantile of $z(X_1, x_2)$ with $X_1 \sim f(x_1|x_2; \beta_0)$. This is a well-known result and we reproduce it for the sake of completeness.

Relevance of Lemma 3: Lemma 3 implies that the tests in (A.10) are constrained weighted average power maximizers. This property has been discussed in Andrews, Moreira, and Stock (2004), Chernozhukov et al. (2009). Lemma 3 combined with Lemma 2c implies that the tests in (A.10)

[11] See Lehmann and Romano (2005) p. 115 for the definition of bounded completeness.

are admissible in the class of all tests.

Lemma 3: Let ϕ^* be defined as in (A.10) and let $c(\cdot;\alpha)$ be measurable. Under Assumptions F1, F2 $\phi^* \in M(w_1)$; that is, ϕ^* minimizes average risk inside the class of α-similar tests.

Proof: Throughout this proof we assume that $\mathbf{X} = \mathbf{X}_1 \times \mathbf{X}_2$. Fubini's theorem (L2a-Step 1) and Theorem 4.3.2 in Lehmann and Casella (1998) implies that $\phi^* \in M(w_1)$ if and only if ϕ^* solves the problem:

$$\min_{\phi \in \mathcal{C}} \quad \int_X (1-\phi(x)) f_1^*(x) dx$$

$$\int_{\mathbf{X}_1(x_2)} \phi(x_1,x_2) f(x_1|x_2) dx_1 = \alpha$$

except, perhaps, for x_2 that belong to a set of measure zero under the marginal density of $h(x_2,\theta)$ for all $\theta \in \Theta_0$. Re-write the objective function as

$$\max_{\phi \in \mathcal{C}} \quad \int_X \phi(x) f_1^*(x) dx.$$

The product structure of $\mathbf{X}$ and the linearity of the integral allows a further expansion of the previous equation:

$$\max_{\phi \in \mathcal{C}} \quad \int_{X_2} \Big(\int_{\mathbf{X}_1} \phi(x_1,x_2) f_1^*(x_1,x_2) dx_1 \Big) dx_2.$$

Note first that the Neyman Pearson Lemma in Ferguson (1967) p. 204 implies that for a fixed x_2 the WAP test $\phi^*(x_1,x_2)$ solves the problem

$$\max_{\phi \in \mathcal{C}} \quad \int_{\mathbf{X}_1} \phi(x_1,x_2) f_1^*(x_1,x_2)\, dx_1$$

subject to

$$\int_{\mathbf{X}_1} \phi(x_1,x_2) f(x_1|x_2) dx_1 = \alpha.$$

except, perhaps, for x_2 that belong to a set of measure zero under every $h(x_2,\theta)$, $\theta \in \mathrm{Bd}\Theta_0$. Hence, to show that $\phi^*(x_1,x_2) \in M(w_1)$ it only remains to prove that $\phi^*(x_1,x_2)$ is measurable. That is, $\phi^*(x_1,x_2) \in \mathcal{C}(\alpha\text{-s})$. Assumption F0 implies that $\phi^*(x_1,x_2)$ is continuous in x_1, for every x_2. Furthermore, since $c(\cdot,\alpha)$ is measurable, then $\phi^*(x_1,x_2)$ is measurable in x_2, for every x_1. Therefore, $\phi^*(x_1,x_2)$ is a Carathéodory function, as defined in Aliprantis and Border (2006), p. 153. Since the sample space $\mathbf{X}$ is separable (by assumption) and metrizable (for it is a subset of a euclidean space), Lemma 4.5.1 in Aliprantis and Border (2006) p. 153 implies $\phi^* : \mathbf{X} \to [0,1]$ is measurable. *Q.E.D.*

A.6. *Continuity of the critical value function*

Measurability is required for the proof of Theorem 1. This subsection provides two sufficient conditions that imply the continuity of $c(\cdot;\alpha)$ (and hence, its measurability).

Let $f(x;\theta)$ denote the statistical model. Consider the following auxiliary assumptions:

Assumption F3: There exists a function $g(\theta)$ such that:

$$f(x;\theta) \le g(\theta) \quad \forall\, x,$$

and $\int_{\theta} g(\theta)dw(\theta) < \infty$.

Assumption F4: $f(x_1|x_2;\beta_0) > 0$ for every (x_1, x_2) and $f(x_1|x_2;\beta_0)$ is continuous in (x_1, x_2).

Assumptions F0, F3, F4 imply that $c(x_2;\alpha)$ is continuous.

Proof: Note first that Assumption F0 implies that $f_w^*(x)$ is sequentially continuous in x. To see this, consider any sequence $x_n \to x$. Assumption F0 i) implies that $f(x_n;\theta) \to f(x;\theta)$ for almost every $\theta \in \Theta$. Since the weight function $w(\theta)$ is assumed to satisfy $f_w^*(x) < \infty$ for every x then:

$$\begin{aligned} \Big| f_w^*(x_n) - f_w^*(x) \Big| &= \Big| \int_{\Theta} f(x_n;\theta)dw(\theta) - \int_{\Theta} f(x;\theta)dw(\theta) \Big| \\ &\leq \int_{\Theta} \Big| f(x_n;\theta) - f(x;\theta) \Big| dw(\theta). \end{aligned}$$

By Assumption F3, the Dominated Convergence Theorem applies and we can conclude that

$$\int_{\Theta} \Big| f(x_n;\theta) - f(x;\theta) \Big| dw(\theta) \to 0.$$

Consequently, $f_w^*(x_n) \to f_w^*(x)$. Furthermore, Assumption F4 implies that the test statistic:

$$z(x_1, x_2) = f_w(x_1, x_2)/f(x_1|x_2;\beta_0)$$

is continuous in (x_1, x_2).

Let $x_{2,n} \to x_2$ and let $X_{1,n} \sim f(x_1|x_{2,n};\beta_0)$. Consider the sequence of random variables.

$$z(X_{1,n}, x_{2,n}).$$

By Scheffe's theorem and the continuity of $f(x_1|x_2;\beta_0)$ at x_2, $X_{1,n} \xrightarrow{d} X \sim f(x_1|x_2;\beta_0)$. Therefore, the random vector $(X_{1,n}, x_{2,n}) \xrightarrow{d} (X, x_2)$. The continuous mapping theorem implies that

$$z(X_{1,n}, x_{2,n}) \xrightarrow{d} z(X_1, x_2).$$

Lemma 21.2 in Van der Vaart (2000) implies $c(x_{2,n};\alpha) \to c(x_2;\alpha)$ for any sequence $x_{2,n} \to x_2$. Hence, the critical value function is continuous and, consequently, measurable.