

SUPPLEMENTARY MATERIAL
Admissible, Similar Tests:
A characterization.
Appendix C.

APPENDIX C: ADDITIONAL RESULTS

C.1. *Asymptotic validity of the test in Result 1*

The test in Result 1 was derived under the assumption that the rotated reduced-form OLS estimators $(S'_n, T'_n)'$ have the exact distribution:

$$Q_{\beta, \pi, \Sigma}^n \equiv \mathcal{N}_{2k} \left(\begin{array}{c} ([b'_0 \otimes \mathbb{I}_k] \Sigma (b_0 \otimes \mathbb{I}_k))^{-1/2} (\beta - \beta_0) \sqrt{n} \pi \\ [(a'_0 \otimes \mathbb{I}_k) \Sigma^{-1} (a_0 \otimes \mathbb{I}_k)]^{-1/2} (a'_0 \otimes \mathbb{I}_k) \Sigma^{-1} (a \otimes \mathbb{I}_k) \sqrt{n} \pi \end{array}, \mathbb{I}_{2k} \right),$$

where Σ is of the form $\Psi \otimes \Phi$. In any finite sample, however, the law of $(S'_n, T'_n)'$ is a function of (β, π) , the sample size, and the joint distribution between the instrumental variables and reduced-form residuals, denoted F . In fact, one can write:

$$\left(\begin{array}{c} ([b'_0 \otimes \mathbb{I}_k] \widehat{\Sigma} (b_0 \otimes \mathbb{I}_k))^{-1/2} (b'_0 \otimes \mathbb{I}_k) \\ [(a'_0 \otimes \mathbb{I}_k) \widehat{\Sigma}^{-1} (a_0 \otimes \mathbb{I}_k)]^{-1/2} (a'_0 \otimes \mathbb{I}_k) \widehat{\Sigma}^{-1} \end{array} \right) \sqrt{n} \widehat{\gamma}_n \sim P_{\beta, \pi, F}^n,$$

where $\widehat{\Sigma}$ is an estimator of the variance of $\sqrt{n} \widehat{\gamma}_n$. This variance depends on F and such dependence is denoted $\Sigma(F)$. The estimator $\widehat{\Sigma}$ need not have the Kronecker form, even when $\Sigma(F)$ does.

If one assumes that for n large enough the distributions P^n and Q^n are ‘close’ to each other (under the null), then one would expect the rate of Type I error computed under P^n to be close to that obtained under Q^n .

PRELIMINARIES: I introduce some additional notation in order to establish Part 2 of Result 1.

1. *Bounded Lipschitz Distance:* Let $d_{\text{BL}}(P, Q) = \sup_{h \in \text{BL}_1} |\mathbb{E}_P[h(X)] - \mathbb{E}_Q[h(X)]|$ denote the Bounded Lipschitz distance between any pair of probability measures P and Q . For definitions and notation, see p. 73, Section 1.12 of [Van der Vaart and Wellner \(1996\)](#). Note also that the Bounded Lipschitz metric is equivalent to the ‘ β' ’ metric between Borel probability measures defined in p. 394 of [Dudley \(2002\)](#).

2. *δ -Expansion of a set A :* For any $\delta > 0$ let A^δ denote the δ -expansion of the set $A \subseteq \mathbb{R}^m$. This is $A^\delta = \{y \in \mathbb{R}^m \mid d(x, y) \leq \delta \text{ for some } x \in A\}$.

3. *A bound on the distance between probability measures:* One can show that for any measurable set A and any $\delta > 0$:

$$(C.1) \quad -Q((A^c)^\delta \setminus A^c) - \frac{1}{\delta} d_{\text{BL}_1}(P, Q) \leq P(A) - Q(A) \leq \frac{1}{\delta} d_{\text{BL}_1}(P, Q) + Q(A^\delta \setminus A),$$

where A^c is the complement of $A \subseteq \mathbb{R}^m$. I use the right-hand side of this inequality to establish the main result.

ASSUMPTION L0: Suppose that the class of distributions \mathcal{F} is such that:

$$\lim_{n \rightarrow \infty} \sup_{(\pi, F) \in \mathbb{R}^k \times \mathcal{F}} d_{\text{BL}}(P_{\beta_0, \pi, F}^n, Q_{\beta_0, \pi, \Sigma(F)}^n) \rightarrow 0.$$

That is, the Bounded Lipschitz distance between the measures $P_{\beta, \pi, F}^n$ and $Q_{\beta, \pi, \Sigma(F)}^n$ converges to zero as the sample size grows large (uniformly over π and F).

ASYMPTOTIC VALIDITY OF THE TEST IN RESULT 1: If Assumption L0 holds and there are constants $0 < \underline{\lambda} < \overline{\lambda} < \infty$ such that the eigenvalues of $\Sigma(F)$ belong to an interval $[\underline{\lambda}, \overline{\lambda}]$ for any

$F \in \mathcal{F}$, then:

$$\limsup_{n \rightarrow \infty} \sup_{(\pi, F) \in \mathbb{R}^k \times \mathcal{F}} P_{\beta_0, \pi, F}^n (z_{\text{WAP}}(S, T) - c_{\text{WAP}}(T, \alpha) > 0) \leq \alpha.$$

This means that the rate of Type I error of the test in Result 1 is uniformly controlled over $(\pi, F) \in \mathbb{R}^k \times \mathcal{F}$.

Consider the test statistic

$$z(S, T) \equiv S'S - T'T + 8 \ln \left(I_0 \left[(1/8) \sqrt{(S'S - T'T)^2 + 4(S'T)^2} \right] \right)$$

and let $c(T; \alpha)$ denote its conditional critical value. I would like to show that if Assumption L0 holds over the class \mathcal{F} , then:

$$(C.2) \quad \limsup_{n \rightarrow \infty} \sup_{(\pi, F) \in \mathbb{R}^k \times \mathcal{F}} P_{\beta_0, \pi, F}^n (z(S, T) - c(T; \alpha) \geq 0) \leq \alpha.$$

I establish the asymptotic validity of the test in Result 1 in 6 steps:

STEP 0: Define

$$A \equiv \{(s', t')' \in \mathbb{R}^{2k} \mid z(s, t) - c(t; \alpha) \geq 0\}.$$

Note immediately that Equation (C.1) implies that for any sample size n and any $(\pi, F) \in \mathbb{R}^k \times \mathcal{F}$:

$$\begin{aligned} P_{\beta_0, \pi, F}^n(A) &\leq Q_{\beta_0, \pi, \Sigma(F)}^n(A) + \frac{1}{\delta} d_{\text{BL}_1}(P_{\beta_0, \pi, F}^n, Q_{\beta_0, \pi, \Sigma(F)}^n) + Q_{\beta_0, \pi, \Sigma(F)}^n(A^\delta \setminus A), \\ &= \alpha + \frac{1}{\delta} d_{\text{BL}_1}(P_{\beta_0, \pi, F}^n, Q_{\beta_0, \pi, \Sigma(F)}^n) + Q_{\beta_0, \pi, \Sigma(F)}^n(A^\delta \setminus A). \end{aligned}$$

where the last equality follows by the definition of the conditional critical value $c(T; \alpha)$. Thus, in order to establish (C.2), I need to show that

$$\frac{1}{\delta} d_{\text{BL}_1}(P_{\beta_0, \pi, F}^n, Q_{\beta_0, \pi, \Sigma(F)}^n) + Q_{\beta_0, \pi, \Sigma(F)}^n(A^\delta \setminus A)$$

can be made arbitrary small, uniformly over the values of (π, F) . By the weak convergence assumption in Part 2 of Result 1, for any fixed δ there is $M_\epsilon(\delta) \in \mathbb{N}$ such that whenever $n \geq M_\epsilon(\delta)$ the term $\delta^{-1} d_{\text{BL}_1}(P_{\beta_0, \pi, F}^n, Q_{\beta_0, \pi, \Sigma(F)}^n)$ can be made smaller than ϵ . Thus, I only need to establish the following result.

GOAL: For every $\epsilon > 0$ there is δ_ϵ and N_ϵ such that for all $n \geq N_\epsilon$

$$\sup_{(\pi, F) \in \mathbb{R}^k \times \mathcal{F}} Q_{\beta_0, \pi, \Sigma(F)}^n(A^{\delta_\epsilon} \setminus A) \leq \epsilon.$$

The proof of this result requires a series of intermediate steps. I exploit the fact that the test statistic $z(s, t)$ satisfies a Lipschitz condition whenever (s, t) is restricted to an appropriate set.

STEP 1: (A bound on $Q_{\beta_0, \pi, \Sigma(F)}^n(A^\delta \setminus A)$): Define the sets

$$(C.3) \quad B(\underline{b}_1, \bar{b}_2) \equiv \{(s', t')' \in \mathbb{R}^{2k} \mid s's \in [\underline{b}_1, \bar{b}_2]\},$$

$$(C.4) \quad C(\underline{c}_1, \bar{c}_2) \equiv \{(s', t')' \in \mathbb{R}^{2k} \mid t't \in [\underline{c}_1, \bar{c}_2]\},$$

$$(C.5) \quad D(\underline{d}_1, \bar{d}_2) \equiv \{(s', t')' \in \mathbb{R}^{2k} \mid (s't)^2/t't \in [\underline{d}_1, \bar{d}_2]\},$$

where $\underline{b}_1, \bar{b}_2, \underline{c}_1, \bar{c}_2, \underline{d}_1, \bar{d}_2$ are positive, finite constants. I want to study the behavior of $A^\delta \setminus A$ inside

and outside the sets defined above. Note that for any n, π, F and δ :

$$\begin{aligned}
Q_{\beta_0, \pi, \Sigma(F)}^n(A^\delta \setminus A) &= Q_{\beta_0, \pi, \Sigma(F)}^n(A^\delta \setminus A \cap [B(\underline{b}_1, \bar{b}_1) \cap C(\underline{c}_1, \bar{c}_2) \cap D(\underline{d}_1, \bar{d}_2)]) \\
&+ Q_{\beta_0, \pi, \Sigma(F)}^n(A^\delta \setminus A \cap [B(\underline{b}_1, \bar{b}_1) \cap C(\underline{c}_1, \bar{c}_2) \cap D(\underline{d}_1, \bar{d}_2)]^c) \\
&\quad \text{(by the additivity property of probability measures)} \\
&\leq Q_{\beta_0, \pi, \Sigma(F)}^n(A^\delta \setminus A \cap [B(\underline{b}_1, \bar{b}_1) \cap C(\underline{c}_1, \bar{c}_2) \cap D(\underline{d}_1, \bar{d}_2)]) \\
&+ Q_{\beta_0, \pi, \Sigma(F)}^n(A^\delta \setminus A \cap B^c(\underline{b}_1, \bar{b}_1)) + Q_{\beta_0, \pi, \Sigma(F)}^n(A^\delta \setminus A \cap C^c(\underline{c}_1, \bar{c}_2)) \\
&+ Q_{\beta_0, \pi, \Sigma(F)}^n(A^\delta \setminus A \cap D(\underline{d}_1, \bar{d}_2)) \\
&\quad \text{(where I have used Boole's inequality)} \\
&\leq Q_{\beta_0, \pi, \Sigma(F)}^n(A^\delta \setminus A \cap [B(\underline{b}_1, \bar{b}_1) \cap C(\underline{c}_1, \bar{c}_2) \cap D(\underline{d}_1, \bar{d}_2)]) \\
&+ Q_{\beta_0, \pi, \Sigma(F)}^n(B^c(\underline{b}_1, \bar{b}_1)) + Q_{\beta_0, \pi, \Sigma(F)}^n(D^c(\underline{d}_1, \bar{d}_2)) \\
&+ Q_{\beta_0, \pi, \Sigma(F)}^n(A^\delta \setminus A \cap C^c(\underline{c}_1, \bar{c}_2)) \\
&\quad \text{(by the monotonicity of probability measures)} \\
&\leq Q_{\beta_0, \pi, \Sigma(F)}^n(A^\delta \setminus A \cap [B(\underline{b}_1, \bar{b}_1) \cap C(\underline{c}_1, \bar{c}_2) \cap D(\underline{d}_1, \bar{d}_2)]) \\
&+ \mathbb{P}(\chi_k^2 \notin (\underline{b}_1, \bar{b}_2)) + \mathbb{P}(\chi_1^2 \notin (\bar{d}_1, \bar{d}_2)) \\
&+ Q_{\beta_0, \pi, \Sigma(F)}^n(A^\delta \setminus A \cap C^c(\underline{c}_1, \bar{c}_2)),
\end{aligned}$$

where the second to last line has uses the fact that under any probability measure $Q_{\beta_0, \pi, \Sigma(F)}^n$:¹²

$$S'_n S_n \stackrel{Q_{\beta_0, \pi, F}^n}{\sim} \chi_k^2 \text{ and } (S'_n T_n)^2 / T'_n T_n \stackrel{Q_{\beta_0, \pi, F}^n}{\sim} \chi_1^2.$$

MAIN CONCLUSION OF STEP 1: I have shown that for any $\delta > 0$ and any positive finite constants $\underline{b}_1, \bar{b}_2, \underline{c}_1, \bar{c}_2, \underline{d}_1, \bar{d}_2$:

$$\begin{aligned}
\text{(C.6)} \quad Q_{\beta_0, \pi, \Sigma(F)}^n(A^\delta \setminus A) &\leq Q_{\beta_0, \pi, \Sigma(F)}^n(A^\delta \setminus A \cap [B(\underline{b}_1, \bar{b}_1) \cap C(\underline{c}_1, \bar{c}_2) \cap D(\underline{d}_1, \bar{d}_2)]) \\
&+ \mathbb{P}(\chi_k^2 \notin (\underline{b}_1, \bar{b}_2)) + \mathbb{P}(\chi_1^2 \notin (\bar{d}_1, \bar{d}_2)) \\
&+ Q_{\beta_0, \pi, \Sigma(F)}^n(A^\delta \setminus A \cap C^c(\underline{c}_1, \bar{c}_2)).
\end{aligned}$$

I now argue that for an appropriate selection of constants, the test statistic $z(s, t)$ and its critical value $c(t; \alpha)$ satisfy a Lipschitz condition when restricted to the set $[B(\underline{b}_1, \bar{b}_1) \cap C(\underline{c}_1, \bar{c}_2) \cap D(\underline{d}_1, \bar{d}_2)]$.

STEP 2—PART A): (Lipschitz property of $z(s, t)$): I show that there exists a constant M_1 —that only depends on the sets B, C, D —such that for any

$$(s'_0, t'_0)', (s'_1, t'_1)' \in \mathcal{K} \equiv [B(\underline{b}_1, \bar{b}_1) \cap C(\underline{c}_1, \bar{c}_2) \cap D(\underline{d}_1, \bar{d}_2)],$$

then:

$$|z(s_0, t_0) - z(s_1, t_1)| < M_1 \|(s'_0, t'_0) - (s'_1, t'_1)\|.$$

To verify the Lipschitz property on $[B(\underline{b}_1, \bar{b}_1) \cap C(\underline{c}_1(\epsilon), \bar{c}_2(\epsilon)) \cap D(\underline{d}_1, \bar{d}_2)]$, it is sufficient to show that, over this set, the derivative of $z(s, t)$ is continuous in its arguments. This observation, together with the fact that $[B(\underline{b}_1, \bar{b}_1) \cap C(\underline{c}_1(\epsilon), \bar{c}_2(\epsilon)) \cap D(\underline{d}_1, \bar{d}_2)]$ is compact gives the desired result. Note

¹²I also use the fact that $\Sigma(F)$ is invertible for any element $F \in \mathcal{F}$.

that the partial derivative of $z(s, t)$ with respect to s is given by:

$$(C.7) \quad z_s(s, t) = 2S + \frac{8I_1 \left((1/8) \sqrt{(s's - t't)^2 + 4t't(s't)^2/t't} \right)}{I_0 \left((1/8) \sqrt{(s's - t't)^2 + 4t't(s't)^2/t't} \right)} \frac{1}{8} \frac{4(s's - t't)s + 8t't((s't)/t't)s}{2\sqrt{(s's - t't)^2 + 4t't((s't)^2/t't)}}$$

$$(C.8) \quad z_t(s, t) = 2t + \frac{8I_1 \left((1/8) \sqrt{(s's - t't)^2 + 4t't(s't)^2/t't} \right)}{I_0 \left((1/8) \sqrt{(s's - t't)^2 + 4t't(s't)^2/t't} \right)} \frac{1}{8} \frac{4(s's - t't)t + 8t't((s't)/t't)t}{2\sqrt{(s's - t't)^2 + 4t't((s't)^2/t't)}}$$

where $I_v(\cdot)$ is the modified Bessel function of the first kind defined in Section 9.6, p. 374 of [Abramowitz and Stegun \(1964\)](#). The formulae above use the fact that the derivative of the modified Bessel function of the first kind of order 0, I_0 , is the modified Bessel function of order 1, I_1 ; see formula 9.6.27 in p. 376 of [Abramowitz and Stegun \(1964\)](#). The continuity of the derivatives and the fact that:

$$\sqrt{(s's - t't)^2 + 4t't(s't)^2/t't}$$

is bounded away from zero over the set

$$[B(\underline{b}_1, \bar{b}_1) \cap C(\underline{c}_1, \bar{c}_2) \cap D(\underline{d}_1, \bar{d}_2)]$$

implies that the Lipschitz condition holds.

STEP 2—PART B): (Lipschitz property of $c(t; \alpha)$): Part *a*) showed that for any selection of constants (b,c,d) the test statistic $z(s, t)$ satisfies the Lipschitz condition when restricted to \mathcal{K} . I now introduce a parameter γ and show that for any given $\gamma > 0$, $\epsilon > 0$ and any pair of constants $\underline{c}_1, \bar{c}_2$, one can find $\underline{b}_1^*, \bar{b}_2^*, \underline{d}_1^*, \bar{d}_2^*$, and M_2 —that depend on $\underline{c}_1, \bar{c}_2, \gamma$ and ϵ —such that for any t_0, t_1 satisfying:

$$(s_0, t_0), (s_1, t_1) \in \mathcal{K}^* \equiv [B(\underline{b}_1^*, \bar{b}_1^*) \cap C(\underline{c}_1, \bar{c}_2) \cap D(\underline{d}_1^*, \bar{d}_2^*)], \quad \text{for some } s_0, s_1 \in \mathbb{R}^k$$

the critical value function satisfies a Lipschitz-type condition:

$$|c(t_0; \alpha) - c(t_1; \alpha)| < M_2 \|t_0 - t_1\| + \gamma/2,$$

and:

$$\mathbb{P}(\chi_k^2 \notin (\underline{b}_1^*, \bar{b}_1^*)) + \mathbb{P}(\chi_1^2 \notin (\underline{d}_1^*, \bar{d}_2^*)) < \epsilon/3.$$

The parameter γ controls how close to is the conditional critical value to satisfy the Lipschitz condition.

To show this, note that for any constant $z \in \mathbb{R}$ and $(\pi, \mathcal{F}) \in \mathbb{R}^k \times \mathcal{F}$:

$$\begin{aligned} Q_{\beta_0, \pi, \Sigma(F)}^n(z(s, t_0) \leq z \mid t = t_0) &= \mathbb{P}(z(S, t_0) \leq z), \quad S \sim \mathcal{N}(0, \mathbb{I}_k) \\ &= \mathbb{P}(z(S, t_0) - z(S, t_1) + z(S, t_1) \leq z) \\ &= \mathbb{P}(z(S, t_1) \leq z - (z(S, t_0) - z(S, t_1)) \text{ and } (S, t_0), (S, t_1) \in \mathcal{K}) \\ &+ \mathbb{P}(z(S, t_1) \leq z - (z(S, t_0) - z(S, t_1)) \text{ and } (S, t_0), (S, t_1) \in \mathcal{K}^c). \end{aligned}$$

where $\mathcal{K} \equiv [B(\underline{b}_1, \bar{b}_1) \cap C(\underline{c}_1, \bar{c}_2) \cap D(\underline{d}_1, \bar{d}_2)]$. Since $z(s, t)$ satisfies the Lipschitz condition in \mathcal{K} with

constant $M_1(\mathcal{K})$ it follows that:

$$\begin{aligned} \mathbb{P}\left(z(S, t_1) \leq z - M_1(\mathcal{K})\|t_0 - t_1\|\right) &\leq \mathbb{P}\left(z(S, t_1) \leq z - M_1(\mathcal{K})\|t_0 - t_1\| \text{ and } (S, t_0), (S, t_1) \in \mathcal{K}\right) \\ &+ \mathbb{P}\left(z(S, t_1) \leq z - M_1(\mathcal{K})\|t_0 - t_1\| \text{ and } (S, t_0), (S, t_1) \in \mathcal{K}^c\right) \\ &\leq \mathbb{P}\left(z(S, t_1) \leq z - (z(S, t_0) - z(S, t_1)) \text{ and } (S, t_0), (S, t_1) \in \mathcal{K}\right) \\ &+ \mathbb{P}\left((S, t_0), (S, t_1) \in \mathcal{K}^c\right), \end{aligned}$$

which implies that:

$$\mathbb{P}\left(z(S, t_1) \leq z - M_1(\mathcal{K})\|t_0 - t_1\|\right) - \mathbb{P}\left((S, t_0), (S, t_1) \in \mathcal{K}^c\right)$$

is less than or equal to

$$\mathbb{P}\left(z(S, t_1) \leq z - (z(S, t_0) - z(S, t_1)) \text{ and } (S, t_0), (S, t_1) \in \mathcal{K}\right).$$

Note also that:

$$\mathbb{P}\left(z(S, t_1) \leq z - (z(S, t_0) - z(S, t_1)) \text{ and } (S, t_0), (S, t_1) \in \mathcal{K}\right) \leq \mathbb{P}\left(z(S, t_1) \leq z + M_1(\mathcal{K})\|t_0 - t_1\|\right).$$

Note now that for any $t \in \mathbb{R}^k$, the critical value function is continuous in α . Therefore, there exists a positive constant, $\eta_\gamma(\underline{c}_1, \bar{c}_2) > 0$, such that for any t such that $t' t \in [\underline{c}_1, \bar{c}_2]$:

$$|c(t; \alpha + \eta_\gamma(\underline{c}_1, \bar{c}_2)) - c(t; \alpha)| \leq \gamma/2, \quad |c(t; \alpha - \eta_\gamma(\underline{c}_1, \bar{c}_2)) - c(t; \alpha)| \leq \gamma/2.$$

Since for any vectors $t_0, t_1 \neq \mathbf{0}_{k \times 1}$:

$$\mathbb{P}\left((S, t_0), (S, t_1) \in \mathcal{K}^c\right) \leq \mathbb{P}\left(\chi_k^2 \notin (\underline{b}_1, \bar{b}_1) + \mathbb{P}\left(\chi_1^2 \notin (\underline{d}_1, \bar{d}_2)\right)\right),$$

one can then choose $0 < \underline{b}_1^* \equiv \underline{b}_1(\eta_\gamma(\underline{c}_1, \bar{c}_2), \epsilon) < \bar{b}_1^* \equiv \bar{b}_1(\eta_\gamma(\underline{c}_1, \bar{c}_2), \epsilon) < \infty$, and $0 < \underline{d}_1^* \equiv \underline{d}_1(\eta_\gamma(\underline{c}_1, \bar{c}_2), \epsilon) < \bar{d}_2^* \equiv \bar{d}_2(\eta_\gamma(\underline{c}_1, \bar{c}_2), \epsilon) < \infty$ such that

$$\mathbb{P}\left(\chi_k^2 \notin (\underline{b}_1^*, \bar{b}_1^*) + \mathbb{P}\left(\chi_1^2 \notin (\underline{d}_1^*, \bar{d}_2^*)\right)\right) < \min\{\eta_\gamma(\underline{c}_1, \bar{c}_2), \epsilon/3\}$$

for any t_0, t_1 . This implies that:

$$(C.9) \quad \mathbb{P}\left(z(S, t_1) \leq z - M_2(\underline{c}_1, \bar{c}_2, \gamma, \epsilon)\|t_0 - t_1\|\right) - \eta_\gamma(\underline{c}_1, \bar{c}_2) \leq \mathbb{P}\left(z(S, t_0) \leq z\right)$$

$$(C.10) \quad \mathbb{P}\left(z(S, t_0) \leq z\right) \leq \mathbb{P}\left(z(S, t_1) \leq z + M_2(\underline{c}_1, \bar{c}_2, \gamma, \epsilon)\|t_0 - t_1\|\right) + \eta_\gamma(\underline{c}_1, \bar{c}_2),$$

where

$$M_2(\underline{c}_1, \bar{c}_2, \gamma, \epsilon) \equiv M_1\left(\underline{b}_1(\eta_\gamma(\underline{c}_1, \bar{c}_2), \epsilon), \bar{b}_1(\eta_\gamma(\underline{c}_1, \bar{c}_2), \epsilon), \underline{c}_1, \bar{c}_2, \underline{d}_1(\eta_\gamma(\underline{c}_1, \bar{c}_2), \epsilon), \bar{d}_2(\eta_\gamma(\underline{c}_1, \bar{c}_2), \epsilon)\right).$$

For simplicity we write M_2 instead of $M_2(\underline{c}_1, \bar{c}_2, \gamma, \epsilon)$ whenever it is convenient.

Since (C.9) holds for any z , in particular it holds for $z = c(t_1; \alpha) + M_2\|t_0 - t_1\|$. Consequently:

$$\begin{aligned} \mathbb{P}\left(z(S, t_0) \leq c(t_1; \alpha) + M_2\|t_0 - t_1\|\right) &\geq \mathbb{P}\left(z(S, t_1) \leq c(t_1; \alpha) - \eta_\gamma(\underline{c}_1, \bar{c}_2)\right) \\ &= 1 - \alpha - \eta_\gamma(\underline{c}_1, \bar{c}_2). \end{aligned}$$

This implies that

$$(C.11) \quad c(t_0; \alpha + \eta_\gamma(\underline{c}_1, \bar{c}_2)) \leq c(t_1; \alpha) + M_2\|t_0 - t_1\|.$$

Likewise, equation (C.10) holds for any z , in particular it holds for $z = c(t_1; \alpha) - M_2||t_0 - t_1||$. This implies that:

$$\begin{aligned} \mathbb{P}(z(S, t_0) \leq c(t_1; \alpha) - M_2||t_0 - t_1||) &\leq \mathbb{P}(z(S, t_1) \leq c(t_1; \alpha)) + \eta_\gamma(\underline{c}_1, \bar{c}_2) \\ &= (1 - \alpha) + \eta_\gamma(\underline{c}_1, \bar{c}_2). \end{aligned}$$

This implies that:

$$(C.12) \quad c(t_0; \alpha - \eta_\gamma(\underline{c}_1, \bar{c}_2)) \geq c(t_1; \alpha) - M_2||t_0 - t_1||.$$

MAIN CONCLUSION OF STEP 2: Finally, (C.11)-(C.12) and the definition of $\eta_\gamma(\underline{c}_1, \bar{c}_2)$ imply that for any $\gamma > 0$, $\epsilon > 0$ and any pair of constants $\underline{c}_1, \bar{c}_2$, one can find $\underline{b}_1^*, \bar{b}_2^*, \underline{d}_1^*, \bar{d}_2^*$, and M_2 —that depend on $\underline{c}_1, \bar{c}_2, \gamma$ and ϵ —such that for any:

$$(s_0, t_0), (s_1, t_1) \in \mathcal{K}^* \equiv [B(\underline{b}_1^*, \bar{b}_1^*) \cap C(\underline{c}_1, \bar{c}_2) \cap D(\underline{d}_1^*, \bar{d}_2^*)],$$

the critical value function satisfies the Lipschitz-type condition:

$$|c(t_0; \alpha) - c(t_1; \alpha)| < M_2||t_0 - t_1|| + \gamma/2.$$

and:

$$\mathbb{P}(\chi_k^2 \notin (\underline{b}_1^*, \bar{b}_1^*) + \mathbb{P}(\chi_1^2 \notin (\underline{d}_1^*, \bar{d}_2^*)) < \epsilon/3.$$

STEP 3: (Exploiting the Lipschitz property to manipulate $A^\delta \setminus A$) The constants in Step 2 depend on $\gamma > 0$, $\epsilon > 0$ and $\underline{c}_1, \bar{c}_2$. This step fixes $\gamma > 0$ and shows how to choose an appropriate enlargement of the set A as a function of γ . The Lipschitz condition established at the end of Step 2 allows for a convenient upper ‘bound’ on the set:

$$(C.13) \quad A^{\delta(\gamma)} \setminus A \cap [B(\underline{b}_1^*, \bar{b}_1^*) \cap C(\underline{c}_1, \bar{c}_2) \cap D(\underline{d}_1^*, \bar{d}_2^*)].$$

In particular, I show that for any $\gamma > 0$, there exists $\delta(\gamma)$ such that:

$$(s', t')' \in A^{\delta(\gamma)} \setminus A \cap [B(\underline{b}_1^*, \bar{b}_1^*) \cap C(\underline{c}_1, \bar{c}_2) \cap D(\underline{d}_1^*, \bar{d}_2^*)]$$

implies that:

$$-\gamma \leq z(s, t) - c(t; \alpha) \leq 0.$$

This inclusion relation is convenient as it allows for the selection of the auxiliary parameter γ to make the probability of the set (C.13) uniformly small over (π, F) .

To establish the desired result, note that $x \equiv (s', t')' \in A^\delta \setminus A$ implies that:

$$z(s, t) - c(t; \alpha) < 0, \text{ (as } x \equiv (s', t')' \notin A),$$

and also that, for any δ , there exists $x_0(\delta) \equiv (s'_{0,\delta}, t'_{0,\delta})' \in A$ such that

$$d(x, x_0(\delta)) \leq \delta.$$

Since the functions $s's, t't, (s't)^2/(t't)$ defining the set:

$$\mathcal{K}^* \equiv [B(\underline{b}_1^*, \bar{b}_1^*) \cap C(\underline{c}_1, \bar{c}_2) \cap D(\underline{d}_1^*, \bar{d}_2^*)],$$

are Lipschitz continuous when restricted to \mathcal{K}^* , there exists δ^* small enough for which the corresponding $x_0(\delta^*)$ belongs to the set \mathcal{K}^* . In this case we have that:

$$\begin{aligned}
\| (z(s, t) - c(t; \alpha)) - (z(s_0, \delta^*, t_0, \delta^*) - c(t_0; \alpha)) \| &\leq \| z(s, t) - z(s_0, \delta^*, t_0, \delta^*) \| + \| c(t; \alpha) - c(t_0, \delta^*; \alpha) \|, \\
&\leq (M_1(\mathcal{K}^*) + M_2(\underline{c}_1, \bar{c}_2, \gamma))d(x, x_0(\delta^*)) + \gamma/2, \\
&\quad (\text{where I have used Step 2 part a) and b}), \\
&\leq (M_1 + M_2)\delta^* + \gamma/2
\end{aligned}$$

Since $x \notin A$ and $x_0(\delta^*) \in A$ implies that

$$0 \geq (z(s, t) - c(t; \alpha)) \geq (z(s, t) - c(t; \alpha)) - (z(s_0, \delta^*, t_0, \delta^*) - c(t_0; \alpha)) \geq -(M_1 + M_2)\delta^* - \gamma/2$$

MAIN CONCLUSION OF STEP 3: Taking $\delta(\gamma) \equiv \min\{\delta^*, \frac{\gamma}{2(M_1+M_2)}\}$ it follows that

$$(s', t')' \in A^{\delta(\gamma)} \setminus A \cap [B(\underline{b}_1^*, \bar{b}_1^*) \cap C(\underline{c}_1, \bar{c}_2) \cap D(\underline{d}_1^*, \bar{d}_2^*)]$$

implies that:

$$-\gamma \leq z(s, t) - c(t; \alpha) \leq 0.$$

I now exploit this relation to show that one can choose γ to guarantee that

$$\sup_{\pi, F \in \mathbb{R}^k \times \mathcal{F}} Q_{\beta_0, \pi, \mathcal{F}}^n(A^{\delta(\gamma)} \setminus A \cap [B(\underline{b}_1^*, \bar{b}_1^*) \cap C(\underline{c}_1, \bar{c}_2) \cap D(\underline{d}_1^*, \bar{d}_2^*)])$$

can be made arbitrarily small.

STEP 4: (Choosing γ as a function of ϵ) Remember that equation (C.6) in Step 1 established that for any $\delta > 0$ and any constants $\underline{b}_1, \bar{b}_1, \underline{c}_1, \bar{c}_2, \underline{d}_1, \bar{d}_2$:

$$\begin{aligned}
Q_{\beta_0, \pi, \Sigma(F)}^n(A^\delta \setminus A) &\leq Q_{\beta_0, \pi, \Sigma(F)}^n(A^\delta \setminus A \cap [B(\underline{b}_1, \bar{b}_1) \cap C(\underline{c}_1, \bar{c}_2) \cap D(\underline{d}_1, \bar{d}_2)]) \\
&\quad + \mathbb{P}(\chi_k^2 \notin (\underline{b}_1, \bar{b}_2)) + \mathbb{P}(\chi_1^2 \notin (\bar{d}_1, \bar{d}_2)) \\
&\quad + Q_{\beta_0, \pi, \Sigma(F)}^n(A^\delta \setminus A \cap C^c(\underline{c}_1, \bar{c}_2)).
\end{aligned}$$

Step 4 showed that for any $\gamma > 0$ there is way of selecting the enlargement parameter $\delta(\gamma) > 0$ and constants $\underline{b}_1^*, \bar{b}_2^*, \underline{d}_1^*, \bar{d}_2^*$ —that depend on $\underline{c}_1, \bar{c}_2$ and γ —such tha the probabilityt:

$$Q_{\beta_0, \pi, \Sigma(F)}^n(A^{\delta(\gamma)} \setminus A \cap [B(\underline{b}_1^*, \bar{b}_1^*) \cap C(\underline{c}_1, \bar{c}_2) \cap D(\underline{d}_1^*, \bar{d}_2^*)])$$

is less than or equal to

$$(C.14) \quad Q_{\beta_0, \pi, \Sigma(F)}^n(-\gamma \leq z(s, t) - c(t; \alpha) \leq 0 \cap [B(\underline{b}_1^*, \bar{b}_1^*) \cap C(\underline{c}_1, \bar{c}_2) \cap D(\underline{d}_1^*, \bar{d}_2^*)])$$

I now show that there exists $\gamma_\epsilon > 0$ small enough such that:

$$Q_{\beta_0, \pi, \Sigma(F)}^n(-\gamma_\epsilon \leq z(s, t) - c(t; \alpha) \leq 0 \cap [B(\underline{b}_1^*, \bar{b}_1^*) \cap C(\underline{c}_1, \bar{c}_2) \cap D(\underline{d}_1^*, \bar{d}_2^*)]) < \epsilon/3$$

for any n, π, F .

To show this, define—for any t such that $t't \in [\underline{c}_1, \bar{c}_2]$ —the function $\gamma_\epsilon(t)$ to satisfy:

$$\mathbb{P}_S(-\gamma_\epsilon(t) \leq g(S, t) \leq 0 \text{ and } S'S \in (\underline{b}_1^*, \bar{b}_1^*) \text{ and } (S't)^2/(t't) \in (\underline{d}_1^*, \bar{d}_1^*)), \quad S \sim \mathcal{N}(0, \mathbb{I}_k),$$

where $g(s, t; \alpha) \equiv z(s, t) - c(t; \alpha)$ and t is treated as fixed vector. Let

$$\gamma_\epsilon \equiv \inf_{\{t \mid t't \in [\underline{c}_1, \bar{c}_2]\}} \gamma_\epsilon(t)$$

and note that $\gamma_\epsilon > 0$ (otherwise, there will be a value t^* for which the distribution of $\mathbb{P}_S(g(S, t^*) =$

0) $> \epsilon/3$). Note that for any n, π, F :

$$Q_{\beta_0, \pi, \Sigma(F)}^n(-\gamma_\epsilon \leq g(s, t; \alpha) \leq 0 \text{ and } s's \in (\underline{b}_1^*, \bar{b}_1^*) \text{ and } t't \in (\underline{c}_1, \bar{c}_2), \text{ and } (s't)^2/t't \in (\underline{d}_1^*, \bar{d}_1^*)),$$

is the same as:

$$\int_{t \in C(\underline{c}_1, \bar{c}_2)} \left(Q_{\beta_0, \pi, \Sigma(F)}^n(-\gamma_\epsilon \leq g(s, t; \alpha) \leq 0 \text{ and } s's \in (\underline{b}_1^*, \bar{b}_1^*) \text{ and } (s't)^2/t't \in (\underline{d}_1^*, \bar{d}_1^*) \mid t \right) d\mathbb{P}_{\beta_0, \pi, F}^n(t),$$

where $\mathbb{P}_{\beta_0, \pi, F}^n$ is the marginal distribution that $Q_{\beta_0, \pi, F}^n$ induces over (t) . Note that (C.15) equals:

$$\int_{t \in C(\underline{c}_1, \bar{c}_2)} \left(\mathbb{P}_S(-\gamma_\epsilon \leq g(S, t) \leq 0 \text{ and } S'S \in (\underline{b}_1^*, \bar{b}_1^*) \text{ and } (S't)^2/(t't) \in (\underline{d}_1^*, \bar{d}_1^*)) \right) d\mathbb{P}_{\beta_0, \pi, F}^n(t),$$

$s|t$ has distribution $\mathcal{N}_k(\mathbf{0}, \mathbb{I}_k)$ for any n, π, F . And this is smaller than or equal:

$$\begin{aligned} & \int_{t \in C(\underline{c}_1, \bar{c}_2)} \left(\mathbb{P}_S(-\gamma_\epsilon(t) \leq g(S, t) \leq 0 \text{ and } S'S \in (\underline{b}_1^*, \bar{b}_1^*) \text{ and } (S't)^2/(t't) \in (\underline{d}_1^*, \bar{d}_1^*)) \right) d\mathbb{P}_{\beta_0, \pi, F}^n(t), \\ &= \mathbb{P}_{\beta_0, \pi, F}^n(t \in C(\underline{c}_1, \bar{c}_2)) \frac{\epsilon}{3} < \frac{\epsilon}{3}, \text{ (by definition of } \gamma_\epsilon(t)). \end{aligned}$$

MAIN CONCLUSION OF STEP 4: This means that for any $\epsilon > 0$ there exists $\gamma_\epsilon > 0$ small enough such that:

$$Q_{\beta_0, \pi, \Sigma(F)}^n(-\gamma_\epsilon \leq z(s, t) - c(t; \alpha) \leq 0 \cap [B(\underline{b}_1^*, \bar{b}_1^*) \cap C(\underline{c}_1, \bar{c}_2) \cap D(\underline{d}_1^*, \bar{d}_1^*)]) < \epsilon/3$$

for any n, π, F .

STEP 5 (CHOOSING \underline{c}_1 AND \underline{c}_2): Step 1 through Step 4 have shown that for any $\epsilon > 0$ there is a constant $\delta_\epsilon \equiv \delta(\gamma_\epsilon)$ and constants $\underline{b}_1^*, \bar{b}_1^*, \underline{d}_1^*, \bar{d}_1^*$ —that depend on $\underline{c}_1, \bar{c}_2$ and ϵ such that for any n, π, F :

$$\begin{aligned} Q_{\beta_0, \pi, \Sigma(F)}^n(A^{\delta_\epsilon} \setminus A) &\leq Q_{\beta_0, \pi, \Sigma(F)}^n(A^{\delta_\epsilon} \setminus A \cap [B(\underline{b}_1^*, \bar{b}_1^*) \cap C(\underline{c}_1, \bar{c}_2) \cap D(\underline{d}_1^*, \bar{d}_1^*)]) \\ &\quad + \mathbb{P}(\chi_k^2 \notin (\underline{b}_1^*, \bar{b}_1^*)) + \mathbb{P}(\chi_1^2 \notin (\bar{d}_1^*, \underline{d}_1^*)) \\ &\quad + Q_{\beta_0, \pi, \Sigma(F)}^n(A^{\delta_\epsilon} \setminus A \cap C^c(\underline{c}_1, \bar{c}_2)). \\ &\leq \frac{2\epsilon}{3} + Q_{\beta_0, \pi, \Sigma(F)}^n(A^{\delta_\epsilon} \setminus A \cap C^c(\underline{c}_1, \bar{c}_2)). \end{aligned}$$

This means that:

$$\sup_{(\pi, F) \in \mathbb{R}^k \times \mathcal{F}} Q_{\beta_0, \pi, \Sigma(F)}^n(A^{\delta_\epsilon} \setminus A) \leq \frac{2\epsilon}{3} + \sup_{(\pi, F) \in \mathbb{R}^k \times \mathcal{F}} Q_{\beta_0, \pi, \Sigma(F)}^n(A^{\delta_\epsilon} \setminus A \cap C^c(\underline{c}_1, \bar{c}_2)).$$

Thus, I only need to show is that \underline{c}_1 and \bar{c}_2 can be chosen to make the second term on the right of the inequality above smaller than $\epsilon/3$. Let λ^* be defined as:

$$\lambda^* \equiv \max_{F \in \mathcal{F}} (a_0' \otimes \mathbb{I}_k) \Sigma(F)^{-1} (a_0 \otimes \mathbb{I}_k)$$

By assumption, there are constants $0 < \underline{\lambda} < \bar{\lambda} < \infty$ such that $\underline{\lambda} < \lambda^* < \bar{\lambda}$. Fix $c^* \in \mathbb{R}^k$ and partition \mathbb{R}^k as follows:

$$\{\pi \in \mathbb{R}^k \mid : n\|\pi\|^2 \lambda^* \leq c^*\} \cup \{\pi \in \mathbb{R}^k \mid : n\|\pi\|^2 \lambda^* > c^*\} \equiv \Pi_1^n(c^*) \cup \Pi_2^n(c^*).$$

Note that

$$\sup_{(\pi, F) \in \mathbb{R}^k \times \mathcal{F}} Q_{\beta_0, \pi, \Sigma(F)}^n(A^{\delta_\epsilon} \setminus A \cap C^c(\underline{c}_1, \bar{c}_2))$$

is smaller than or equal than the sum of:

$$(C.15) \quad \sup_{(\pi, F) \in \Pi_1^n(c^*) \times \mathcal{F}} Q_{\beta_0, \pi, \Sigma(F)}^n(A^{\delta_\epsilon} \setminus A \cap C^c(\underline{c}_1, \bar{c}_2)),$$

and

$$(C.16) \quad \sup_{(\pi, F) \in \Pi_2^n(c^*) \times \mathcal{F}} Q_{\beta_0, \pi, \Sigma(F)}^n(A^{\delta_\epsilon} \setminus A \cap C^c(\underline{c}_1, \bar{c}_2))$$

STEP 5—PART A): First, I bound the term (C.15). Let $\chi_k^2(c)$ denote a non-central chi-square with k degrees of freedom and centrality parameter c . Note that:

$$\begin{aligned} \sup_{(\pi, F) \in \Pi_1^n(c^*) \times \mathcal{F}} Q_{\beta_0, \pi, \Sigma(F)}^n(A^{\delta_\epsilon} \setminus A \cap C^c(\underline{c}_1, \bar{c}_2)) &\leq \sup_{(\pi, F) \in \Pi_1^n(c^*) \times \mathcal{F}} Q_{\beta_0, \pi, \Sigma(F)}^n(C^c(\underline{c}_1, \bar{c}_2)), \\ &= \sup_{(\pi, F) \in \Pi_1^n(c^*) \times \mathcal{F}} Q_{\beta_0, \pi, \Sigma(F)}^n(t't \notin (\underline{c}_1, \bar{c}_2)), \\ &= \sup_{(\pi, F) \in \Pi_1^n(c^*) \times \mathcal{F}} \mathbb{P}\left(\chi_k^2(n\|\pi\|^2\lambda(F)) \notin (\underline{c}_1, \bar{c}_2)\right), \\ &\quad (\text{where } \lambda(F) \equiv (a'_0 \otimes \mathbb{I}_k)\Sigma(F)^{-1}(a_0 \otimes \mathbb{I}_k)). \end{aligned}$$

Therefore, one can choose constants $\underline{c}_1^*, \bar{c}_2^*$ that depend on c^* and ϵ (but do not depend on the sample size) such that for any $\underline{c}_1 < \underline{c}_1^*$ and $\bar{c}_2 > \bar{c}_2^*$:

$$\sup_{(\pi, F) \in \Pi_1^n(c^*) \times \mathcal{F}} Q_{\beta_0, \pi, \Sigma(F)}^n(A^{\delta_\epsilon} \setminus A \cap C^c(\underline{c}_1, \bar{c}_2)) < \frac{\epsilon}{6}.$$

STEP 5—PART B): Now, I bound the term (C.16). To do this, choose \bar{e} to satisfy

$$\mathbb{P}(\chi_k^2 > \bar{e}) < \frac{\epsilon}{12}.$$

Since this selection of \bar{e} implies that

$$\sup_{(\pi, F) \in \Pi_2^n(c^*) \times \mathcal{F}} Q_{\beta_0, \pi, \Sigma(F)}^n(A^{\delta_\epsilon} \setminus A \cap s's > \bar{e} \cap C^c(\underline{c}_1, \bar{c}_2)) < \frac{\epsilon}{12},$$

it is sufficient to show that there is \bar{c}_2 large enough such that:

$$\sup_{(\pi, F) \in \Pi_2^n(c^*) \times \mathcal{F}} Q_{\beta_0, \pi, \Sigma(F)}^n(A^{\delta_\epsilon} \setminus A \cap s's \leq \bar{e} \cap C^c(\underline{c}_1, \bar{c}_2)) < \frac{\epsilon}{12}$$

The key to establish this result is to show that the for $t't$ large enough and $s's$ in a compact set, the test statistic $z(s, t)$ defined in Result 1 is close to the statistic $(s't)^2/(t't)$.

STEP 5—PART C): Let $\mathfrak{o}(t't)$ denote the function:

$$\mathfrak{o}(t't) \equiv \left(((s's/t't) - 1)^2 + 4(s't)^2/(t't)^2 \right)^{1/2} - 1.$$

Note that:

$$\begin{aligned}
 & 8 \ln \left[I_0 \left((t't/8)(1 + \mathbf{o}(t't)) \right) \right], \\
 = & 8 \ln \left[\frac{e^{(t't/8)(1 + \mathbf{o}(t't))}}{\sqrt{2\pi i((t't/8)(1 + \mathbf{o}(t't)))}} \left(1 + O \left(\frac{1}{(t't/8)(1 + \mathbf{o}(t't))} \right) \right) \right], \\
 & \text{(where I have used the asymptotic approximation} \\
 & \text{for } I_0(z) \text{ in p. 435 of } \text{Olver (1997)} \text{ and the definition} \\
 & \text{of } \sim \text{ in p. 4 of the same book)} \\
 = & 8 \ln \left[\frac{e^{(t't/8)(1 + \mathbf{o}(t't))}}{\sqrt{2\pi i((t't/8)(1 + \mathbf{o}(t't)))}} \right] \\
 + & 8 \ln \left[\left(1 + O \left(\frac{1}{(t't/8)(1 + \mathbf{o}(t't))} \right) \right) \right], \\
 = & t't(1 + \mathbf{o}(t't)) - 4 \ln(2\pi i) - 4 \ln(t't/8) - 4 \ln(1 + \mathbf{o}(t't)) \\
 + & 8 \ln \left[\left(1 + O \left(\frac{1}{(t't/8)(1 + \mathbf{o}(t't))} \right) \right) \right].
 \end{aligned}$$

Therefore, $z_{\text{WAP}}(s, t)$ in Result 1:

$$(s's - t't) + 8 \ln \left[I_0 \left(\frac{1}{8} \left[(s's - t't)^2 + 4(s't)^2 \right]^{1/2} \right) \right] + 4 \ln(2\pi i) + 4 \ln((1/8)t't)$$

can be written in terms of the Conditional Likelihood Ratio statistic (CLR) as follows:

$$\begin{aligned}
 z_{\text{WAP}}(s, t) & \equiv (s's - t't) + t't(1 + \mathbf{o}(t't)) - 4 \ln(1 + \mathbf{o}(t't)) \\
 & + 8 \ln \left[\left(1 + O \left(\frac{1}{(t't/8)(1 + \mathbf{o}(t't))} \right) \right) \right], \\
 & = 2\text{CLR}(s, t) - 4 \ln(1 + \mathbf{o}(t't)) \\
 & + 8 \ln \left[\left(1 + O \left(\frac{1}{(t't/8)(1 + \mathbf{o}(t't))} \right) \right) \right], \\
 & \text{(where we have used the fact that } t't(1 + \mathbf{o}(t't)) \\
 & \text{equals } [(s's - t't)^2 + (s't)^2]^{1/2} \text{).}
 \end{aligned}$$

It is well-known for large values of $t't$ and for values of $s's$ in a compact set the CLR can be approximated by the LM statistic ($\equiv s't/t't$) uniformly over the values of s . Choose $\zeta^* > 0$ to satisfy:

$$\mathbb{P}(-\eta^* \leq N(0, 1) \leq \eta^*) = \frac{\epsilon}{24}.$$

Therefore, using the same argument as in part b) of step 2 one can show for $\zeta^* > 0$ there is \bar{c}_2^* —that depends on ζ^* —such that uniformly over $s's \leq \bar{e}$

$$|z_{\text{WAP}}(s, t) - c_{\text{WAP}}(t; \alpha) - 2LM(s, t) + 2\chi_{1, 1-\alpha}^2| < \zeta^*,$$

where $\chi_{1, 1-\alpha}^2$ is the $1-\alpha$ quantile of a chi-squared random variable with one degree of freedom and $c_{\text{WAP}}(t; \alpha)$ is the conditional critical value of $z_{\text{WAP}}(s, t)$.

STEP 5—PART D): Note that

$$x \equiv (s, t) \in (A^{\delta_\epsilon} \setminus A \cap s's \leq \bar{e} \cap t't > \bar{c}_2^*),$$

implies that:

$$z(s, t) - c(t; \alpha) < 0,$$

and also that there is $x_0(\delta_\epsilon) \equiv (s_0, t_0) \in A$ such that $z(s_0, t_0) - c(t; \alpha) > 0$ and $d(x, x_0(\delta_\epsilon)) < \delta_\epsilon$. Since the test based on the test statistic $z(s, t)$ with conditional critical value $c(t; \alpha)$ is equivalent to the test based on $z_{\text{WAP}}(s, t)$ and $c_{\text{WAP}}(t; \alpha)$, it follows that:

$$z_{\text{WAP}}(s, t) - c_{\text{WAP}}(t; \alpha) < 0, \text{ and } z_{\text{WAP}}(s_0, t_0) - c_{\text{WAP}}(t; \alpha) > 0.$$

Consequently:

$$\text{LM}(s, t) - \chi_{1,1-\alpha}^2 < \zeta^*/2$$

and

$$\text{LM}(s_0, t_0) - \chi_{1,1-\alpha}^2 > -\zeta^*/2.$$

Note that the LM statistic can be written as a function of (S, ω_t) where $\omega_t \equiv t/||t||$. Since the partial derivatives of $(s'\omega_t)^2$ are bounded whenever $s's \leq \bar{e}$, the LM statistic satisfies the Lipschitz condition when $s's$ belongs to the desired domain. Let $M(\bar{e})$ denote the Lipschitz constant of the LM statistic. Since:

$$-d(x, x_0(\delta_\epsilon))M(\bar{e}) \leq \text{LM}(s, t) - \text{LM}(s_0, t_0) \leq M(\bar{e})d(x, x_0(\delta_\epsilon)),$$

then:

$$\begin{aligned} -\delta_\epsilon M(\bar{e}) &\leq -d(x, x_0(\delta_\epsilon))M(\bar{e}) \\ &\leq \text{LM}(s, t) - \chi_{1,1-\alpha}^2 + \chi_{1,1-\alpha}^2 - \text{LM}(s_0, t_0), \\ &\leq \text{LM}(s, t) - \chi_{1,1-\alpha}^2 + \zeta^*/2, \\ &\quad (\text{where I have used the fact that } \text{LM}(s_0, t_0) - \chi_{1,1-\alpha}^2 > -\zeta^*/2), \\ &\leq \zeta^*, \\ &\quad (\text{where I have used the fact that } \text{LM}(s, t) - \chi_{1,1-\alpha}^2 < \zeta^*/2). \end{aligned}$$

One can further shrink δ_ϵ to satisfy $\delta_\epsilon M(\bar{e}) < -\zeta^*$. This means that:

$$\begin{aligned} \sup_{(\pi, F) \in \Pi_2^n(c^*) \times \mathcal{F}} Q_{\beta_0, \pi, \Sigma(F)}^n(A^{\delta_\epsilon} \setminus A \cap s's \leq \bar{e} \cap t't > \bar{c}_2^*) &\leq \sup_{(\pi, F) \in \Pi_2^n(c^*) \times \mathcal{F}} Q_{\beta_0, \pi, \Sigma(F)}^n(-\eta^* \leq \text{LM}(s, t) \leq \eta^*) \\ &= \mathbb{P}(-\eta^* \leq N(0, 1) \leq \eta^*) \\ &= \frac{\epsilon}{24} \end{aligned}$$

Since:

$$\sup_{(\pi, F) \in \Pi_2^n(c^*) \times \mathcal{F}} Q_{\beta_0, \pi, \Sigma(F)}^n(A^{\delta_\epsilon} \setminus A \cap s's \leq \bar{e} \cap C^c(\underline{c}_1, \bar{c}_2^*)),$$

is smaller than or equal:

$$\sup_{(\pi, F) \in \Pi_2^n(c^*) \times \mathcal{F}} Q_{\beta_0, \pi, \Sigma(F)}^n(A^{\delta_\epsilon} \setminus A \cap s's \leq \bar{e} \cap t't \leq \underline{c}_1),$$

plus

$$\sup_{(\pi, F) \in \Pi_2^n(c^*) \times \mathcal{F}} Q_{\beta_0, \pi, \Sigma(F)}^n(A^{\delta_\epsilon} \setminus A \cap s's \leq \bar{e} \cap t't > \underline{c}_2^*),$$

it follows that:

$$\sup_{(\pi, F) \in \Pi_2^n(c^*) \times \mathcal{F}} Q_{\beta_0, \pi, \Sigma(F)}^n(A^{\delta\epsilon} \setminus A \cap s' s \leq \bar{e} \cap C^c(\underline{c}_1, \bar{c}_2^*)) < \frac{\epsilon}{24} + \frac{\epsilon}{24} = \frac{\epsilon}{12}.$$

This implies that:

$$\sup_{(\pi, F) \in \Pi_2^n(c^*) \times \mathcal{F}} Q_{\beta_0, \pi, \Sigma(F)}^n(A^{\delta\epsilon} \setminus A \cap C^c(\underline{c}_1, \bar{c}_2)) < \frac{\epsilon}{12} + \frac{\epsilon}{12} = \frac{\epsilon}{6}.$$

MAIN CONCLUSION OF STEP 5: The conclusion of Step 5 is that there are constants $\underline{c}_1^*, \bar{c}_2^*$ such that:

$$\sup_{(\pi, F) \in \mathbb{R}^k \times \mathcal{F}} Q_{\beta_0, \pi, \Sigma(F)}^n(A^{\delta\epsilon} \setminus A) \leq \frac{2\epsilon}{3} + \sup_{(\pi, F) \in \mathbb{R}^k \times \mathcal{F}} Q_{\beta_0, \pi, \Sigma(F)}^n(A^{\delta\epsilon} \setminus A \cap C^c(\underline{c}_1^*, \bar{c}_2^*)) \leq \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

STEP 1 TO STEP 5: I have shown that for every $\epsilon > 0$ one can choose constants $\underline{b}_1^*, \bar{b}_2^*, \underline{c}_1^*, \bar{c}_2^*, \underline{d}_1^*, \bar{d}_2^*$ such that for any sample size and $(\pi, F) \in \mathbb{R}^k \times \mathcal{F}$:

$$\begin{aligned} Q_{\beta_0, \pi, \Sigma(F)}^n(A^{\delta\epsilon} \setminus A) &\leq Q_{\beta_0, \pi, \Sigma(F)}^n(A^\delta \setminus A \cap [B(\underline{b}_1^*, \bar{b}_1^*) \cap C(\underline{c}_1^*, \bar{c}_2^*) \cap D(\underline{d}_1^*, \bar{d}_2^*)]) \\ &\quad + \mathbb{P}(\chi_k^2 \notin (\underline{b}_1^*, \bar{b}_2^*)) + \mathbb{P}(\chi_1^2 \notin (\bar{d}_1^*, \bar{d}_2^*)) \\ &\quad + Q_{\beta_0, \pi, \Sigma(F)}^n(A^\delta \setminus A \cap C^c(\underline{c}_1^*, \bar{c}_2^*)) \\ &\leq \epsilon \end{aligned}$$

Since for any $\delta > 0$:

$$\begin{aligned} P_{\beta_0, \pi, \mathcal{F}}^n(A) &\leq Q_{\beta_0, \pi, \Sigma(F)}^n(A) + \frac{1}{\delta} d_{\text{BL}_1}(P_{\beta_0, \pi, F}^n, Q_{\beta_0, \pi, \Sigma(F)}^n) + Q_{\beta_0, \pi, \Sigma(F)}^n(A^\delta \setminus A), \\ &= \alpha + \frac{1}{\delta} d_{\text{BL}_1}(P_{\beta_0, \pi, F}^n, Q_{\beta_0, \pi, \Sigma(F)}^n) + Q_{\beta_0, \pi, \Sigma(F)}^n(A^\delta \setminus A), \end{aligned}$$

and, by assumption, there is $M_\epsilon \in \mathbb{N}$ such that for any $n \geq M_\epsilon$:

$$d_{\text{BL}_1}(P_{\beta_0, \pi, F}^n, Q_{\beta_0, \pi, \Sigma(F)}^n) \leq \epsilon \delta_\epsilon,$$

I conclude that for $n \geq M_\epsilon$:

$$\sup_{(\pi, F) \in \mathbb{R}^k \times \mathcal{F}} P_{\beta_0, \pi, \mathcal{F}}^n(A) \leq \alpha + 2\epsilon.$$

This establishes the asymptotic validity of the test in Result 1.

FINAL COMMENTS: The asymptotic validity of the test in Result 1 exploited specific properties of the test statistic $z_{\text{WAP}}(s, t)$. For example, I showed that $z_{\text{WAP}}(s, t)$ is a Lipschitz function function of (s, t) provided $s's, t't$, and $(s't)^2/t't$ are bounded from below and from above. Under the same domain restriction, the proof also uses the existence of a threshold $\gamma_\epsilon > 0$ such that the probability of the event $-\gamma_\epsilon \leq z_{\text{WAP}}(S, T) - c_{\text{WAP}}(T, \alpha) \leq 0$ is smaller than ϵ , uniformly over the family of distributions $Q_{\beta_0, \pi, F}^n$. Finally, the proof used an asymptotic approximation of the modified Bessel function of the first kind for large values of its argument that appears in [Olver \(1997\)](#), p. 435. Under this approximation, the $z_{\text{WAP}}(s, t)$ is approximately $2CLR(s, t)$ when $t't$ is large.

C.2. Local Asymptotic Power of the test in Result 1

Now I derive the local asymptotic power of the test in Result 1 under the following assumption:

ASSUMPTION L1: The class of distributions \mathcal{F} is such that:

$$\lim_{n \rightarrow \infty} d_{\text{BL}} \left(P_{\beta_0 + \frac{c}{\sqrt{n}}, \pi, F}^n, Q_{\beta_0 + \frac{c}{\sqrt{n}}}^n \right) \rightarrow 0$$

This is, the Bounded Lipschitz distance between the measures $P_{\beta, \pi, F}^n$ and $Q_{\beta, \pi, \Sigma(F)}^n$ converges to zero as the sample size grows large for any local alternative of the form $\beta_0 + c/\sqrt{n}$.

ASYMPTOTIC EFFICIENCY OF THE TEST IN RESULT 1: Suppose that Assumption L1 holds and suppose that there are constants $0 < \underline{\lambda} < \bar{\lambda} < \infty$ such that the eigenvalues of $\Sigma(F)$ belong to an interval $[\underline{\lambda}, \bar{\lambda}]$ for any $F \in \mathcal{F}$. If $\Sigma(F) = \Psi(F) \otimes \Phi(F)$ and $\pi \neq \mathbf{0}_{k \times 1}$, then:

$$\liminf_{n \rightarrow \infty} P_{\beta_0 + \frac{c}{\sqrt{n}}, \pi, F}^n(z_{\text{WAP}}(s, t) - c_{\text{WAP}}(t; \alpha) \geq 0) \geq \mathbb{P} \left(\chi_1^2(\mu^2(\beta_0, c, \pi, F)) > \chi_{1, 1-\alpha}^2 \right),$$

where $\chi_1^2(\mu^2(\beta_0, c, \pi, F))$ is a non-central chi-square distribution with centrality parameter:

$$\mu^2(\beta_0, c, \pi, F) \equiv c^2 (\pi' \Phi(F)^{-1} \pi) (b_0' \Psi(F) b_0)^{-1}.$$

I establish the Local efficiency of the test in Result 1 in 6 steps:

STEP 0: Fix $(\pi, F) \in \mathbb{R}^k \times \mathcal{F}$ and suppose that $\|\pi\| \neq 0$. Let \bar{e} denote a positive scalar. In Part c) of Step 5 I have shown that for any $\zeta > 0$ there is $\underline{e}(\zeta)$ such that for any $t't > \underline{e}(\zeta)$ —and uniformly over the values of $s's < \bar{e}$ —it follows that:

$$|z_{\text{WAP}}(s, t) - c_{\text{WAP}}(t; \alpha) - 2\text{LM}(s, t) + 2\chi_{1, 1-\alpha}^2| < \zeta/2,$$

where $\text{LM}(s, t) \equiv (s't)^2/t't$. This means that if $s's < \bar{e}$ and $t't > \underline{e}(\zeta)$:

$$\text{LM}(s, t) - \chi_{1, 1-\alpha}^2 > \zeta/4 \implies z_{\text{WAP}}(s, t) - c_{\text{WAP}}(t; \alpha) \geq 0.$$

Therefore, for any local alternative $\beta(c) \equiv \beta_0 + \frac{c}{\sqrt{n}}$:

$$\begin{aligned} P_{\beta(c), \pi, F}^n(z_{\text{WAP}}(s, t) - c_{\text{WAP}}(t; \alpha) \geq 0) &\geq P_{\beta(c), \pi, F}^n(\text{LM}(s, t) \geq \chi_{1-\alpha}^2 + \zeta \text{ and } s's < \bar{e} \text{ and } t't > \underline{e}(\zeta)), \\ &\geq P_{\beta(c), \pi, F}^n(\text{LM}(s, t) \geq \chi_{1-\alpha}^2 + \zeta/4) \\ &\quad + P_{\beta(c), \pi, F}^n(s's < \bar{e}) \\ &\quad + P_{\beta(c), \pi, F}^n(t't > \underline{e}(\zeta)) - 2, \\ &\quad (\text{where I have used } P(A \cap B) = P(A) + P(B) - P(A \cup B) \\ &\quad \text{twice, and also the fact that } P(A \cup B) \leq 1). \end{aligned}$$

I now characterize the asymptotic behavior of the terms:

$$P_{\beta(c), \pi, F}^n(\text{LM}(s, t) \geq \chi_{1-\alpha}^2 + \zeta/4), \quad P_{\beta(c), \pi, F}^n(s's < \bar{e}), \quad P_{\beta(c), \pi, F}^n(t't > \underline{e}(\zeta)).$$

STEP 1: Consider first the term:

$$P_{\beta(c), \pi, F}^n(\text{LM}(s, t) \geq \chi_{1-\alpha}^2 + \zeta/4).$$

Note that the event:

$$E_1 = \{(s', t')' \in \mathbb{R}^{2k} \mid \text{LM}(s, t) > \chi_{1, 1-\alpha}^2 + \zeta/4\},$$

is the same as the event:

$$E_1 = \{(s', t')' \in \mathbb{R}^{2k} \mid LM(s, t/\sqrt{n}) > \chi_{1,1-\alpha}^2 + \zeta/4\},$$

as $LM(s, t) = (s't)^2/t't = s't/\sqrt{n})^2/(t/\sqrt{n})'(t/\sqrt{n})$.

Let $\tilde{P}_{\beta(c), \pi, F}^n$ denote the distribution of the random vector $(S', (t/\sqrt{n})')'$. Since the transformation $(x, y) \rightarrow (x, y/\sqrt{n})$ is Lipschitz for any $n \in \mathbb{N}$ (with constant 1) it follows—by assumption—that as $n \rightarrow \infty$:¹³

$$d_{\text{BL}_1}(\tilde{P}_{\beta(c), \pi, F}^n, \tilde{Q}_{\beta(c), \pi, \Sigma(F)}^n) \rightarrow 0,$$

where \tilde{Q} is the distribution of:

$$\tilde{Q}_{\beta, \pi, \Sigma(F)}^n \equiv \mathcal{N}_{2k} \left(\begin{array}{c} ([b'_0 \otimes \mathbb{I}_k] \Sigma(b_0 \otimes \mathbb{I}_k))^{-1/2} c\pi \\ [(a'_0 \otimes \mathbb{I}_k) \Sigma(F)^{-1} (a_0 \otimes \mathbb{I}_k)]^{-1/2} (a'_0 \otimes \mathbb{I}_k) \Sigma(F)^{-1} (a_n(c) \otimes \mathbb{I}_k) \pi \end{array}, \begin{pmatrix} \mathbb{I}_k & \mathbf{0}_{k \times k} \\ \mathbf{0}_{k \times k} & \mathbb{I}_k/\sqrt{n} \end{pmatrix} \right),$$

with $a_n(c) = (\beta_0 + c/\sqrt{n}, 1)'$. Moreover, the convergence of the mean vector of this distribution and its covariance matrix this implies that:

$$d_{\text{BL}_1}(\tilde{Q}_{\beta(c), \pi, \Sigma(F)}^n, Q_{\beta_0, \pi, \Sigma(F)}) \rightarrow 0$$

where

$$Q_{\beta_0, \pi, \Sigma(F)} \equiv \mathcal{N}_{2k} \left(\begin{array}{c} ([b'_0 \otimes \mathbb{I}_k] \Sigma(b_0 \otimes \mathbb{I}_k))^{-1/2} c\pi \\ [(a'_0 \otimes \mathbb{I}_k) \Sigma(F)^{-1} (a_0 \otimes \mathbb{I}_k)]^{1/2} \pi \end{array}, \begin{pmatrix} \mathbb{I}_k & \mathbf{0}_{k \times k} \\ \mathbf{0}_{k \times k} & \mathbf{0}_{k \times k} \end{pmatrix} \right).$$

Thus, one can conclude that

$$d_{\text{BL}_1}(\tilde{P}_{\beta(c), \pi, \Sigma(F)}^n, Q_{\beta_0, \pi, \Sigma(F)}) \rightarrow 0.$$

For any event, and in particular for E_1 ,

$$\tilde{P}_{\beta(c), \pi, F}^n(E_1) \geq Q_{\beta_0, \pi, \Sigma(F)}(E_1) - \frac{1}{\delta} d_{\text{BL}_1}(\tilde{P}_{\beta(c), \pi, F}^n, Q_{\beta_0, \pi, \Sigma(F)}) - Q_{\beta_0, \pi, \Sigma(F)}((E_1^c)^\delta \setminus E_1^c)$$

Under the probability measure $Q_{\beta_0, \pi, \Sigma(F)}$ (which does not depend on n), the topological boundary of E_1^c

$$\text{Bd}(E_1^c) \equiv \{(s', t')' \in \mathbb{R}^{2k} \mid LM(s, t) = \chi_{1,1-\alpha}^2 + \zeta/4\},$$

has probability zero. Therefore, there exists $\delta_{\epsilon, \zeta}$ (independent of n) such that:

$$Q_{\beta_0, \pi, F}((E^c)_{\delta_{\epsilon, \zeta}} \setminus E_1^c) < \frac{\epsilon}{6}.$$

Moreover, by choosing $N(\epsilon, \zeta) \in \mathbb{N}$ to be such that for $n \geq N(\epsilon, \zeta)$:

$$d_{\text{BL}_1}(\tilde{P}_{\beta(c), \pi, \Sigma(F)}^n, Q_{\beta_0, \pi, \Sigma(F)}) \leq \delta_{\epsilon, \zeta} \frac{\epsilon}{6},$$

¹³To see this, note that for any function $h \in \text{BL}_1$ we have that:

$$\left| \mathbb{E}_{\tilde{P}_{\beta(c), \pi, F}^n} [h(X)] - \mathbb{E}_{\tilde{Q}_{\beta(c), \pi, F}^n} [h(X)] \right| = \left| \mathbb{E}_{P_{\beta(c), \pi, F}^n} [h \circ g_n(S, T)] - \mathbb{E}_{Q_{\beta(c), \pi, F}^n} [h \circ g_n(S, T)] \right|,$$

where $g_n(s, t) = (s', (t/\sqrt{n})')'$ is an element of BL_1 . Consequently,

$$0 \leq d_{\text{BL}_1}(\tilde{P}_{\beta(c), \pi, F}^n, \tilde{Q}_{\beta(c), \pi, \Sigma(F)}^n) \leq d_{\text{BL}_1}(P_{\beta(c), \pi, F}^n, Q_{\beta(c), \pi, \Sigma(F)}^n).$$

we have that:

$$\begin{aligned}
P_{\beta(c),\pi,F}^n(E_1) &= \tilde{F}_{\beta(c),\pi,F}^n(E_1), \\
&\geq Q_{\beta_0,\pi,\Sigma(F)}(E_1) - \frac{1}{\delta_{\epsilon,\zeta}} d_{\text{BL}_1}(\tilde{F}_{\beta(c),\pi,F}^n, Q_{\beta_0,\pi,\Sigma(F)}) - Q_{\beta_0,\pi,\Sigma(F)}((E_1^c)^{\delta_{\epsilon,\zeta}} \setminus E_1^c), \\
&\geq Q_{\beta_0,\pi,\Sigma(F)}(E_1) - \frac{\epsilon}{6} - \frac{\epsilon}{6}, \\
&= Q_{\beta_0,\pi,\Sigma(F)}(E_1) - \frac{\epsilon}{3}.
\end{aligned}$$

Moreover, under the probability measure $Q_{\beta_0,\pi,\Sigma(F)}(E_1)$:

$$\frac{t'S}{\sqrt{t't}} \sim \mathcal{N}(\mu(\beta_0, c, \pi, F), 1),$$

where

$$\mu(\beta_0, c, \pi, F) \equiv c \frac{\pi' \left([(a'_0 \otimes \mathbb{I}_k)] \Sigma(F)^{-1} (a_0 \otimes \mathbb{I}_k) \right)^{1/2} [(b'_0 \otimes \mathbb{I}_k) \Sigma(F) (b_0 \otimes \mathbb{I}_k)]^{-1/2} \pi}{\left(\pi' [(a'_0 \otimes \mathbb{I}_k)] \Sigma^{-1}(F) (a_0 \otimes \mathbb{I}_k) \pi \right)^{1/2}}.$$

This means that $Q_{\beta_0,\pi,\Sigma(F)}(E_1)$ is the same as:

$$\mathbb{P} \left(\chi_1^2(\mu^2(\beta_0, c, \pi, F)) > \chi_{1,1-\alpha}^2 + \eta \right).$$

This means that for any $\zeta > 0, \epsilon > 0$ there is $N(\epsilon, \zeta) \in \mathbb{N}$ such that for $n \geq N(\epsilon, \zeta)$:

$$P_{\beta(c),\pi,F}^n(E_1) \geq \mathbb{P} \left(\chi_1^2(\mu^2(\beta_0, c, \pi, F)) > \chi_{1,1-\alpha}^2 + \zeta \right) - \frac{\epsilon}{3},$$

where $\chi_1^2(\mu^2(c, \beta_0, c, \pi, F))$ is a non-central chi-square distribution with centrality parameter:

$$\mu(\beta_0, c, \pi, F) \equiv c \frac{\pi' \left([(a'_0 \otimes \mathbb{I}_k)] \Sigma(F)^{-1} (a_0 \otimes \mathbb{I}_k) \right)^{1/2} [(b'_0 \otimes \mathbb{I}_k) \Sigma(F) (b_0 \otimes \mathbb{I}_k)]^{-1/2} \pi}{\left(\pi' [(a'_0 \otimes \mathbb{I}_k)] \Sigma^{-1}(F) (a_0 \otimes \mathbb{I}_k) \pi \right)^{1/2}}.$$

STEP 2: Now I take care of the term $P_{\beta(c),\pi,F}^n(s' s < \bar{\epsilon})$. In particular, I show that $\bar{\epsilon}$ to make this term to be at least $1 - \epsilon/3$ for n large enough. Let E_2 denote the event:

$$E_2 \equiv \{(s', t')' \mid s' s < \bar{\epsilon}\}$$

Note that for any $\delta > 0$:

$$\begin{aligned}
P_{\beta(c),\pi,F}^n(E_2) &\geq Q_{\beta(c),\pi,\Sigma(F)}^n(E_2) - \frac{1}{\delta} d_{\text{BL}_1}(P_{\beta(c),\pi,F}^n, Q_{\beta(c),\pi,\Sigma(F)}^n) - Q_{\beta(c),\pi,\Sigma(F)}^n((E_2^c)^{\delta} \setminus E_2^c) \\
&= \mathbb{P}_S(E_2) - \frac{1}{\delta} d_{\text{BL}_1}(P_{\beta(c),\pi,F}^n, Q_{\beta(c),\pi,\Sigma(F)}^n) - \mathbb{P}_S((E_2^c)^{\delta} \setminus E_2^c),
\end{aligned}$$

where $S \sim \mathcal{N}_k([b'_0 \otimes \mathbb{I}_k] \Sigma(F) (b_0 \otimes \mathbb{I}_k)^{-1/2} \pi c, \mathbb{I}_{2k})$. Once again, since the topological boundary of E_2 has zero measure under \mathbb{P}_S there exists $\delta_{\epsilon, \bar{\epsilon}}$ such that:

$$\mathbb{P}_S((E_2^c)^{\delta_{\epsilon, \bar{\epsilon}}} \setminus E_2^c) < \frac{\epsilon}{9}.$$

This means that one can choose $N(\epsilon, \bar{\epsilon}) \in \mathbb{N}$ such that for $n \geq N(\epsilon, \bar{\epsilon})$:

$$P_{\beta(c),\pi,F}^n(E_2) \geq \mathbb{P}_S(E_2) - \frac{2\epsilon}{9}.$$

Moreover, there is $\bar{\epsilon}^*$ large enough such that $\mathbb{P}_S(E_2) \geq 1 - \frac{\epsilon}{9}$. Therefore, for $n \geq N(\epsilon, \bar{\epsilon}^*)$

$$P_{\beta(c), \pi, F}^n(E_2) \geq 1 - \frac{\epsilon}{3}.$$

STEP 3: Finally, consider the term $P_{\beta(c), \pi, F}^n(t't > \underline{\epsilon})$. I show that for any fixed $\bar{\epsilon}$ there is n large enough such that this term is at least $1 - \epsilon/3$. Define:

$$E_{3,n} \equiv \{(s', t')' \mid t't > \underline{\epsilon}/\sqrt{n}\}.$$

Note that:

$$\begin{aligned} P_{\beta(c), \pi, F}^n(t't > \underline{\epsilon}) &= \tilde{P}_{\beta(c), \pi, F}^n(E_{3,n}), \\ &= Q_{\beta_0, \pi, \Sigma(F)}(E_{3,n}) - \frac{1}{\delta} d_{\text{BL}_1}(\tilde{P}_{\beta(c), \pi, F}^n, Q_{\beta_0, \pi, \Sigma(F)}) - Q_{\beta_0, \pi, \Sigma(F)}((E_{3,n}^c)^\delta \setminus E_{3,n}^c). \\ &\geq Q_{\beta_0, \pi, \Sigma(F)}(E_{3,n}) - \frac{1}{\delta} d_{\text{BL}_1}(\tilde{P}_{\beta(c), \pi, F}^n, Q_{\beta_0, \pi, \Sigma(F)}) - Q_{\beta_0, \pi, \Sigma(F)}((E_{3,n}^c)^\delta). \end{aligned}$$

Under $Q_{\beta_0, \pi, \Sigma(F)}$, the statistic t has a degenerate distribution with all of its mass at $t^* \equiv [(a'_0 \otimes \mathbb{I}_k)]\Sigma^{-1}(F)(a_0 \otimes \mathbb{I}_k)]^{1/2}\pi \equiv 0$. This means that for n large enough:

$$Q_{\beta_0, \pi, \Sigma(F)}(E_{3,n}) = \mathbf{1}\{t^* \in E_{3,n}\} = 1.$$

Take any δ such the ball of radius δ^* around t^* , $B_{\delta^*}(t^*)$, that excludes the origin. Note that for any n such that $\epsilon/\sqrt{n} < \delta^*$:

$$B_{\epsilon^{1/2}/n^{1/4}}(\mathbf{0}_k \times \mathbf{1}) \cap B_{\delta^*}(t^*) = \emptyset.$$

This means that for n large enough there is no $t_0 \in \mathbb{R}^k$ such that: $t'_0 t_0 < \epsilon/\sqrt{n}$ and $d(t_0, t^*) \leq \delta^*$. This means that for n large enough $t^* \notin (E_{3,n}^c)^\delta$. Therefore for n large enough:

$$\begin{aligned} P_{\beta(c), \pi, F}^n(t't > \underline{\epsilon}) &\geq 1 - \frac{1}{\delta^*} d_{\text{BL}_1}(\tilde{P}_{\beta(c), \pi, F}^n, Q_{\beta_0, \pi, \Sigma(F)}) - 0 \\ &\geq 1 - \frac{\epsilon}{3}. \end{aligned}$$

STEP 4: I have shown that:

$$\begin{aligned} P_{\beta(c), \pi, F}^n(z(s, t) - c(t; \alpha) \geq 0) &\geq P_{\beta(c), \pi, F}^n(LM(s, t) \geq \chi_{1-\alpha}^2 + \zeta/4) \\ &\quad + P_{\beta(c), \pi, F}^n(s's < \bar{\epsilon}^*) \\ &\quad + P_{\beta(c), \pi, F}^n(t't > \underline{\epsilon}(\zeta)) - 2. \end{aligned}$$

Step 1, 2, 3 of this proof have established the existence of $N(\epsilon, \zeta, \bar{\epsilon}^*)$ such that for any $n \geq N(\epsilon, \zeta, \bar{\epsilon}^*)$:

$$\begin{aligned} P_{\beta(c), \pi, F}^n(LM(s, t) \geq \chi_{1-\alpha}^2 + \zeta/4) &\geq \mathbb{P}\left(\chi_1^2(\mu^2(\beta_0, c, \pi, F)) > \chi_{1, 1-\alpha}^2 + \zeta\right) - \frac{\epsilon}{3}, \\ P_{\beta(c), \pi, F}^n(s's < \bar{\epsilon}^*) &\geq 1 - \frac{\epsilon}{3}, \\ P_{\beta(c), \pi, F}^n(t't > \underline{\epsilon}(\zeta)) &\geq 1 - \frac{\epsilon}{3}, \end{aligned}$$

where the centrality parameter $\mu^2(\beta_0, c, \pi, F)$ is given by:

$$\mu^2(\beta_0, c, \pi, F) \equiv \left(\frac{\pi' \left([(a'_0 \otimes \mathbb{I}_k)]\Sigma(F)^{-1}(a_0 \otimes \mathbb{I}_k)]^{1/2} [(b'_0 \otimes \mathbb{I}_k)]\Sigma(F)(b_0 \otimes \mathbb{I}_k)]^{-1/2} \right) \pi}{\left(\pi' [(a'_0 \otimes \mathbb{I}_k)]\Sigma^{-1}(F)(a_0 \otimes \mathbb{I}_k)] \pi \right)^{1/2}} \right)^2.$$

This implies that:

$$\liminf_{n \rightarrow \infty} P_{\beta_0 + \frac{c}{\sqrt{n}}, \pi, F}^n(z_{\text{WAP}}(s, t) - c_{\text{WAP}}(t; \alpha) \geq 0) \geq \mathbb{P}\left(\chi_1^2(\mu^2(\beta_0, c, \pi, F)) > \chi_{1, 1-\alpha}^2 + \zeta\right),$$

for any $\zeta > 0$. Since the distribution on the right-hand side is continuous in ζ , then:

$$\liminf_{n \rightarrow \infty} P_{\beta_0 + \frac{c}{\sqrt{n}}, \pi, F}^n(z_{\text{WAP}}(s, t) - c_{\text{WAP}}(t; \alpha) \geq 0) \geq \mathbb{P}\left(\chi_1^2(\mu^2(\beta_0, c, \pi, F)) > \chi_{1, 1-\alpha}^2\right)$$

STEP 5: If $\Sigma(F) = \Psi(F) \otimes \Phi(F)$:

$$[(a'_0 \otimes \mathbb{I}_k)]\Sigma(F)^{-1}(a_0 \otimes \mathbb{I}_k) = (a'_0 \Psi^{-1} a_0) \Phi^{-1}$$

and

$$[(b'_0 \otimes \mathbb{I}_k)]\Sigma(F)(b_0 \otimes \mathbb{I}_k) = (b'_0 \Psi b_0) \Phi.$$

Therefore, the expression for the centrality parameter simplifies to:

$$\mu^2(\beta_0, c, \pi, F) \equiv c^2(\pi' \Phi^{-1} \pi)(b'_0 \Psi b_0)^{-1}.$$

FINAL COMMENTS: The lower bound on local power above implies that the test in Result 1 is as powerful (locally) as a GMM-Wald test for β_0 based on the sample moment condition:

$$\frac{1}{\sqrt{n}} Z'(y - \beta_0 x) = \mathbf{0}.$$

To see this, note that under Assumption L0 the asymptotic variance of the sample moment condition is simply QW_0Q where Q is the probability limit of $Z'Z/n$ and $W_0 \equiv (b'_0 \otimes \mathbb{I}_k)\Sigma(b_0 \otimes \mathbb{I}_k)$. Therefore, the efficient GMM estimator for β (assuming W_0 is known) is:

$$\begin{aligned} \beta_{\text{GMM}} &= \left(X'Z(Z'ZW_0Z'Z)^{-1}Z'X\right)^{-1}X'Z(Z'ZW_0Z'Z)^{-1}Z'y \\ &= \left(\widehat{\gamma}_2 W_0^{-1} \widehat{\gamma}_2\right)^{-1} \widehat{\gamma}_2 W_0^{-1} \widehat{\gamma}_1. \end{aligned}$$

The efficient α -level GMM-Wald test for $\beta = \beta_0$ rejects whenever:

$$\left(\left(\widehat{\gamma}_2 W_0^{-1} \widehat{\gamma}_2\right)^{-1/2} \widehat{\gamma}_2 W_0^{-1} \sqrt{n}(\widehat{\gamma}_1 - \beta_0 \widehat{\gamma}_2)\right)^2 > \chi_{1, 1-\alpha}^2,$$

and this test has local power, under alternatives of the form $\beta_0 + c/\sqrt{n}$, given by:

$$\mathbb{P}\left(\chi_1^2(c^2(\pi' W_0^{-1} \pi)) > \chi_{1, 1-\alpha}^2\right).$$

If Σ is of the form $\Psi \otimes \Omega$, then $c^2(\pi' W_0^{-1} \pi)$ coincides with the centrality parameter $\mu^2(\beta_0, c, \pi, F)$.