Supplementary Material
Admissible, Similar Tests:
A characterization.
Appendix B.

APPENDIX B: KRONECKER CASE

B.1. *Weights for (β, π) in the Kronecker case*

This section analyzes the properties of the weights:

$$\begin{pmatrix}\beta\pi \\ \pi\end{pmatrix} = n^{-1/2}\left(\Psi_\Sigma \widetilde{C}_0' \otimes \Phi_\Sigma^{1/2}\right)\rho(\phi \otimes \omega),\ \widetilde{C}_0 \equiv \begin{pmatrix}(b_0'\Psi_\Sigma b_0)^{-1/2} b_0' \\ (a_0'\Psi_\Sigma^{-1} a_0)^{-1/2} a_0'\Psi_\Sigma^{-1}\end{pmatrix}, \tag{B.1}$$

with

$$\phi \sim \mathcal{U}(\mathcal{S}^1), \quad \omega \sim \mathcal{U}(\mathcal{S}^{k-1}), \tag{B.2}$$

and

$$\rho|\phi,\omega \sim \sqrt{\chi_k^2}\ /\ (\phi' \otimes \omega')\left(\widetilde{C}_0 \Psi_\Sigma' \otimes {\Phi_\Sigma'}^{1/2}\right)\Sigma^{-1}\left(\Psi_\Sigma \widetilde{C}_0' \otimes \Phi_\Sigma^{1/2}\right)(\phi \otimes \omega). \tag{B.3}$$

The main assumption of this section is that $\Sigma = \Psi \otimes \Phi$. Note first that when $\Sigma = \Psi \otimes \Phi$:

$$\Psi_\Sigma = (\text{vec}(\Phi)'\text{vec}(\Phi))^{1/2}\Psi, \quad \Phi_\Sigma = \Phi/(\text{vec}(\Phi)'\text{vec}(\Phi))^{1/2}.$$

Therefore, we can write the weights in (B.1) as:

$$\begin{pmatrix}\beta\pi \\ \pi\end{pmatrix} = n^{-1/2}\left(\Psi C_0' \otimes \Phi\right)\rho(\phi \otimes \omega),\ C_0 \equiv \begin{pmatrix}(b_0'\Psi b_0)^{-1/2} b_0' \\ (a_0'\Psi^{-1} a_0)^{-1/2} a_0'\Psi^{-1}\end{pmatrix}, \tag{B.4}$$

Weight for β: Under (B.4), the parameter β equals:

$$\beta = \frac{[1,0]\Psi C_0'\phi}{[0,1]\Psi C_0'\phi},$$

This ratio can be simplified using the following equalities. First:

$$\begin{aligned}
[1,0]\Psi C_0'\phi &= [1,0]\Psi\left[b_0(b_0'\Psi b_0)^{-1/2},\ \Psi^{-1}a_0(a_0'\Psi^{-1}a_0)^{-1/2}\right]\phi, \\
&\quad \text{(by definition of } C_0\text{)} \\
&= \left[[1,0]\Psi b_0(b_0'\Psi b_0)^{-1/2},\ \beta_0(a_0'\Psi^{-1}a_0)^{-1/2}\right]\phi, \\
&\quad \text{(as } [1,0]a_0 = \beta_0\text{)}.
\end{aligned}$$

Second:

$$a_0'\Psi^{-1}a_0 = \frac{1}{\det(\Psi)}a_0'\begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix}\Psi\begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix}a_0 = \det\left(\begin{pmatrix}1 & -\beta_0 \\ 0 & 1\end{pmatrix}\Psi\begin{pmatrix}1 & -\beta_0 \\ 0 & 1\end{pmatrix}'\right)^{-1} b_0'\Psi b_0,$$

and

$$1 - r(\beta_0)^2 = \det\left(\begin{pmatrix}1 & -\beta_0 \\ 0 & 1\end{pmatrix}\Psi\begin{pmatrix}1 & -\beta_0 \\ 0 & 1\end{pmatrix}'\right)\Big/(b_0'\Psi b_0)([0,1]\Psi[0,1]'),$$

where $r(\beta_0)$ refers to the correlation coefficient of $(b_0';[0,1])\Psi(b_0';[0,1])'$. This means that the numerator for β is given by:

$$\begin{aligned}
[1,0]\Psi C_0'\phi &= [1,0]\Psi b_0(b_0'\Psi b_0)^{-1/2}\phi_1 + \beta_0\sqrt{1-r^2(\beta_0)}([0,1]\Psi[0,1]')^{1/2}\phi_2, \\
&= [1,0]\Psi b_0(b_0'\Psi b_0)^{-1/2}\phi_1 - \beta_0([0,1]\Psi b_0)(b_0'\Psi b_0)^{-1/2}\phi_1 \\
&+ \beta_0\left(r(\beta_0)\phi_1 + \sqrt{1-r^2(\beta_0)}([0,1]\phi_2\right)([0,1]\Psi[0,1]')^{1/2}, \\
&\quad \text{(where I have added and subtracted } \beta_0 r(\beta_0)([0,1]\Psi[0,1]')^{1/2}\phi_1\text{)}.
\end{aligned}$$

Therefore:

$$[1,0]\Psi C_0'\phi = (b_0'\Psi b_0)^{1/2}\phi_1 + \beta_0[0,1]\Psi C_0'\phi,$$

where I have used the fact that:

$$[0,1]\Psi C_0'\phi = \left(r(\beta_0)\phi_1 + \sqrt{1-r^2(\beta_0)}([0,1]\phi_2\right)([0,1]\Psi[0,1]')^{1/2}.$$

This means that β can be written as:

$$\beta = \frac{[1,0]\Psi C_0'\phi}{[0,1]\Psi C_0'\phi} = \frac{(b_0'\Psi b_0)^{1/2}\phi_1}{[0,1]\Psi C_0'\phi} + \beta_0. \tag{B.5}$$

Weight for π: The distribution of the first-stage coefficient is given by:

$$\sqrt{n}\pi = ([0,1]\Psi C_0'\phi)\Phi^{1/2}\rho\omega, \tag{B.6}$$

This means that:

$$\sqrt{n}\pi \mid \phi \sim \mathcal{N}_k\left(\mathbf{0}\,,\ ([0,1]\Psi C_0'\phi)^2\Phi\right).$$

Comparison to the MM2 weights: I claim that if $\Sigma = \Psi \otimes \Phi$ the weights in (3.5) and (3.6) are equivalent to the 'MM2' weights proposed in MM15. To see this, note that (B.5) implies that the vector $(\beta,1)'$ can be written as $\Psi C_0'\phi$ divided by $[0,1]\Psi C_0'\phi$. Since the vector $(c_\beta,d_\beta)'$ in MM15 equals $C_0(\beta,1)'$, then:

$$||(c_\beta,d_\beta)'|| = ||C_0(\beta,1)'|| = ||C_0\Psi C_0'\phi||/|[0,1]\Psi C_0'\phi| = 1/|[0,1]\Psi C_0'\phi|.$$

Therefore,

$$1/||(c_\beta,d_\beta)'||^2 = ([0,1]\Psi C_0'\phi)^2.$$

This implies that

$$\sqrt{n}\pi \mid \phi \sim \mathcal{N}_k\left(\mathbf{0}\,,\ (||(c_\beta,d_\beta)'||^{-2}\Phi\right),$$

which is the same distribution as in MM15, up to a scaling constant. Also, MM15 assumes that the distribution of the angle of

$$C_0(\beta,1)'/||C_0(\beta,1)'||$$

is uniform on $[-\pi,\pi]$. Under (B.5) it follows that

$$C_0(\beta,1)'/||C_0(\beta,1)'|| = \phi,$$

where ϕ is uniformly distributed on the unit circle $\mathcal{S}^1$. Part ii) of exercise 5.2.4 in Stroock (1999) implies that ϕ can be written as $[\cos(\theta)',\sin(\theta)']'$ where θ is uniformly distributed on a connected interval of length $2pi$.

Distribution of $\sqrt{\lambda}(\beta-\beta_0)$ and λ: The Monte-Carlo exercises in Andrews et al. (2006) depend on the parameters:

$$\lambda \equiv n\pi\Phi^{-1}\pi, \text{ and } \sqrt{\lambda}(\beta-\beta_0).$$

Equation (B.6) implies that:

$$\begin{aligned}
\lambda &\equiv \left(([0,1]\Psi C_0'\phi)\Phi^{1/2}\rho\omega\right)\Phi^{-1}\left(([0,1]\Psi C_0'\phi)\Phi^{1/2}\rho\omega\right)\\
&= ([0,1]\Psi C_0'\phi)^2\rho^2\omega'\Phi^{1/2}\Phi^{-1}\Phi^{1/2}\omega\\
&= ([0,1]\Psi C_0'\phi)^2\rho^2.
\end{aligned}$$

Consequently, equation (B.5) implies that

$$\begin{aligned}\sqrt{\lambda}(\beta - \beta_0) &= \sqrt{([0,1]\Psi C_0'\phi)^2\rho^2}\left(\frac{(b_0'\Phi b_0)^{1/2}\phi_1}{[0,1]\Psi C_0'\phi}\right)\\ &= (b_0'\Phi b_0)^{1/2}\rho\phi_1.\end{aligned}$$

Therefore:

$$\text{(B.7)} \qquad \sqrt{\lambda}(\beta - \beta_0) = \left(b_0'\Psi b_0\right)^{1/2}\rho\phi_1,$$

$$\text{(B.8)} \qquad \lambda = ([0,1]\Psi C_0'\phi)^2\rho^2,$$

The probability density function of $(\sqrt{\lambda}(\beta - \beta_0), \lambda)$ is given in Figure 3 in the main text of the paper.

B.2. *Proof or Result 1:*

Preliminaries: The statistical model under consideration is:

$$\widehat{\gamma}_n \sim \mathcal{N}_{2k}\left(\begin{pmatrix}\beta\pi\\ \pi\end{pmatrix}, \Sigma/\sqrt{n}\right), \quad \text{where } \widehat{\gamma} \equiv \begin{pmatrix}(Z'Z)^{-1}Z'y\\ (Z'Z)^{-1}Z'x\end{pmatrix}. \tag{B.9}$$

By assumption

$$\Sigma = \Psi \otimes \Phi,$$

where Ψ is a matrix of dimension 2×2 and Φ is matrix of dimension $k\times k$ (both positive definite and symmetric). The model in (B.9) is transformed into a Gaussian location model with independent components by rotating the reduced-form parameters:

$$\begin{pmatrix} S_n\\ T_n\end{pmatrix} \equiv$$

$$\begin{bmatrix}([b_0'\otimes\mathbb{I}_k)\Sigma(b_0\otimes\mathbb{I}_k)]^{-1/2} & \mathbf{0}\\ \mathbf{0} & [(a_0'\otimes\mathbb{I}_k)\Sigma^{-1}(a_0\otimes\mathbb{I}_k)]^{-1/2}\end{bmatrix}\begin{bmatrix}(b_0'\otimes\mathbb{I}_k)\\ (a_0'\otimes\mathbb{I}_k)\Sigma^{-1}\end{bmatrix}\sqrt{n}\widehat{\gamma}_n.$$

In the Kronecker case, this transformation can be written as:

$$\left(C_0\otimes\Phi^{-1/2}\right)\sqrt{n}\widehat{\gamma}_n,$$

where

$$C_0 \equiv \begin{pmatrix}(b_0'\Psi b_0)^{-1/2}b_0'\\ \\ (a_0'\Psi^{-1}a_0)^{-1/2}a_0'\Psi^{-1}\end{pmatrix}, \quad b_0=[1,-\beta_0]' \quad a_0=[\beta_0,1]'.$$

Consequently, the rotated statistical model becomes:

$$\begin{pmatrix} S_n\\ T_n\end{pmatrix} \sim \mathcal{N}_{2k}\left(\left(C_0\otimes\Phi^{-1/2}\right)\sqrt{n}\begin{pmatrix}\beta\pi\\ \pi\end{pmatrix}, \ \mathbb{I}_{2k}\right) \tag{B.10}$$

In order to compute the WAP similar test, I need to integrate (B.10) with respect to the weights in (3.4). In the Kronecker case, these weights are given by:

$$\begin{pmatrix}\beta\pi\\ \pi\end{pmatrix} = n^{-1/2}\left(\Psi C_0'\otimes\Phi^{1/2}\right)\rho(\phi\otimes\omega),$$

where (ϕ,ω,ρ) are independent random variables with the following distributions:

$$\phi\sim\mathcal{U}(\mathcal{S}^1),\ \omega\sim\mathcal{U}(\mathcal{S}^k),\ \rho\sim\sqrt{\chi^2_k}.$$

Thus, integrating (B.10) with respect to the weights in (3.4) is equivalent to integrating the likelihood of:

$$\begin{pmatrix} S\\ T\end{pmatrix} \sim \mathcal{N}_{2k}\left(\rho\left(\phi\otimes\omega\right)\ ,\ \mathbb{I}_{2k}\right),$$

with respect to the weights for (ρ,ϕ,ω).

Derivation of the Integrated Likelihoods: Let $f(S,T;\rho,\phi,\omega)$ denote the Gaussian statistical model for (S,T) given parameters (ρ,ϕ,ω). This is:

$$f(S,T;\rho,\phi,\omega) = c_1\exp\left(-\frac{1}{2}([S',T']'-\rho(\phi\otimes\omega))'([S',T']'-\rho(\phi\otimes\omega))\right),$$

where c_1 is a non-negative constant. The function $f(S,T;\rho,\phi,\omega)$ is thus analogous to $f(x;\theta)$ in Section 2 of the paper.

Step 1: (Integrate ω) Note that:

$$\begin{aligned}\widetilde{f}(S,T;\rho,\phi) &\equiv c_2\int_{\mathcal{S}^{k-1}} f(S,T;\rho,\phi,\omega)d\lambda_{\mathcal{S}^{k-1}}(\omega)\\ &= a_2(Q)\exp\Big(-\rho^2/2\Big)\int_{\mathcal{S}^{k-1}}\exp\Big(([S,T]\phi)'\rho\omega\Big)d\lambda_{\mathcal{S}^{k-1}}(\omega)\end{aligned}$$

where $\lambda_{\mathcal{S}^{k-1}}(\cdot)$ is the uniform measure over the $k-1$ dimensional sphere $\mathcal{S}^{k-1}$ defined in Chamberlain (2007) and Stroock (1999). In addition,

$$a_2(Q) \equiv c_2\exp\Big(-\frac{1}{2}[S'S+T'T]\Big)$$

c_2 is a non-negative constant.

Step 2: (Integrate ρ) By assumption $\rho\sim\sqrt{\chi^2_k}$ independently of ϕ and ω. The latter implies that the density of ρ is given by:

$$m_1(\rho)\equiv\frac{1}{2^{k/2}\Gamma(k/2)}(\rho^2)^{(k/2)-1}e^{-(\rho^2/2)}2\rho$$

Note that using Fubini's Theorem and the change of variables formula:

$$\int_{\mathbb{R}+}\widetilde{f}(S,T;\rho,\phi)m_1(\rho)d\rho$$

$$\begin{aligned}&= a_2(Q)\int_{\mathbb{R}+}\Big(\exp(-\rho^2/2)\int_{\mathcal{S}^{k-1}}\exp\Big(\Big[(S,T)\rho\phi\Big]'\boldsymbol{\omega}\Big)d\lambda_{\mathcal{S}^{k-1}}(\boldsymbol{\omega})\Big)m_1(\rho)d\rho\\ &= a_2(Q)\int_{\mathcal{S}^{k-1}}\Big(\int_{\mathbb{R}_+}\exp\Big(\Big[(S,T)\rho\phi\Big]'\boldsymbol{\omega}\Big)m_1(\rho)\exp(-\rho^2/2)d\rho\Big)d\lambda_{\mathcal{S}^{k-1}}(\boldsymbol{\omega})\\ &= a_3(Q)\int_{\mathcal{S}^{k-1}}\Big(\int_{\mathbb{R}_+}\exp\Big(\Big[(S,T)\rho\phi\Big]'\boldsymbol{\omega}\Big)\exp(-\rho^2)\rho^{k-1}dp\Big)d\lambda_{\mathcal{S}^{k-1}}(\boldsymbol{\omega})\\ &= a_3(Q)\int_{\mathcal{S}^{k-1}}\Big(\int_{\mathbb{R}_+}\exp\Big(\Big[(S,T)\phi\Big]'\rho\omega\Big)\exp(-(\rho\boldsymbol{\omega})'(\rho\boldsymbol{\omega}))\rho^{k-1}dp\Big)d\lambda_{\mathcal{S}^{k-1}}(\boldsymbol{\omega})\end{aligned}$$

where the last line follows from $\boldsymbol{\omega}'\boldsymbol{\omega}=1$ and $a_3(Q)=a_2(Q)2/(2^{k/2}\Gamma(k/2))$. Theorem 5.2.2, p. 86 in Stroock (1999) implies:

$$\begin{aligned}\int_{\mathbb{R}+}\widetilde{f}(S,T;\rho,\phi)m_1(\rho)d\rho &= a_3(Q)\int_{\mathbb{R}^K}\exp\Big(\Big[(S,T)\phi\Big]'x\Big)\exp(-x'x)dx\\ &\qquad\text{(by applying Theorem 5.2.2 to the function }\exp\Big(\Big[(S,T)\phi\Big]'x\Big)\exp(-x'x))\\ &= a_4(Q)\exp\Big(\frac{1}{4}\phi'Q\phi\Big),\ Q\equiv[S,T]'[S,T],\end{aligned}$$

where the last inequality follows by definition of the moment generating function of a k-dimensional multivariate normal evaluated at $(S,T)\phi$. Note that $a_4(Q)\equiv(2pi)^{k/2}a_3(Q)$.

Step 3: (Integrate ϕ) Part ii) of exercise 5.2.4 in p. 87 of Stroock (1999) implies that

$$\int_{\mathcal{S}^1} a_4(Q)\exp\left(\frac{1}{4}\phi' Q\phi\right) d\lambda_{\mathcal{S}^1}(\phi) = \frac{a_4(Q)}{2pi}\int_0^{2pi}\exp\left(\frac{1}{4}[\cos(\theta),\sin(\theta)]Q[\cos(\theta),\sin(\theta)]'\right)d\theta \equiv f^*(S,T).$$

Note that the largest and smallest eigenvalue of the matrix Q are given by:

$$\zeta_{max} = \frac{1}{2}\left[(S'S+T'T)+\sqrt{(S'S-T'T)^2+4(S'T)^2}\right]$$
$$\zeta_{min} = \frac{1}{2}\left[(S'S+T'T)-\sqrt{(S'S-T'T)^2+4(S'T)^2}\right]$$

The following step derives the numerator of the WAP-similar test in Result 1.

Step 3aux: We show that:

$$f^*(S,T) \quad = \quad a_4(Q)\exp\left(\frac{1}{4}\Big(\zeta_{\max}+\zeta_{\min}\Big)\right) I_0\left(\frac{1}{4}(\zeta_{max}-\zeta_{min})\right),$$

where $I_0(\cdot)$ is the modified Bessel function of the first kind, defined in Abramowitz and Stegun (1964), Section 9.6, p. 375, and

$$a_4(Q) \propto (2pi)^{k/2} a_2(Q) 2/(2^{k/2}\Gamma(k/2))\exp\Big(-\frac{1}{2}[S'S+T'T]\Big).$$

Proof: Let $L \equiv S'S - \zeta_{min}$. Note that L is the Likelihood Ratio Statistic as defined in Andrews et al. (2006) p. 722. The eigenvector associated to largest eigenvalue of the matrix Q equals:

$$e_{max} \equiv (L, S'T)'/\sqrt{L^2+(S'T)^2}$$

Define $\widehat{\theta} \in [0, 2pi]$ implicitly by the following equation:

$$[\cos(\widehat{\theta}),\sin(\widehat{\theta})]' = e_{max}$$

Therefore,

$$P \equiv \begin{pmatrix} \cos(\widehat{\theta}) & \sin(\widehat{\theta}) \\ \sin(\widehat{\theta}) & -\cos(\widehat{\theta}) \end{pmatrix}$$

yields the spectral decomposition of the matrix Q; that is:

$$P\begin{pmatrix} \zeta_{max} & 0 \\ 0 & \zeta_{min} \end{pmatrix} P' = Q.$$

Note that for any θ:

$$P'\begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} = \begin{pmatrix} \cos(\widehat{\theta})\cos(\theta)+\sin(\widehat{\theta})\sin(\theta) \\ \sin(\widehat{\theta})\cos(\theta)-\cos(\widehat{\theta})\sin(\theta) \end{pmatrix} = \begin{pmatrix} \cos(\widehat{\theta}-\theta) \\ \sin(\widehat{\theta}-\theta) \end{pmatrix}$$

Therefore:

$$
\begin{aligned}
f^*(S,T) &= \frac{a_4(Q)}{2pi}\int_0^{2pi} \exp\Big(\frac{1}{4}\Big[\zeta_{max}\cos^2(\widehat{\theta}-\theta)+\zeta_{min}\sin^2(\widehat{\theta}-\theta)\Big]\Big)d\theta \\
&= \frac{a_4(Q)}{2pi}\int_{\widehat{\theta}-2pi}^{\widehat{\theta}} \exp\Big(\frac{1}{4}\Big[\zeta_{max}\cos^2(\theta)+\zeta_{min}\sin^2(\theta)\Big]\Big)d\theta \\
&\qquad \text{(where he have changed the integration variable)} \\
&= \exp\Big(\frac{1}{4}\zeta_{min}\Big)\frac{a_4(Q)}{2pi}\int_{\widehat{\theta}-2pi}^{\widehat{\theta}} \exp\Big(\frac{1}{4}\Big[(\zeta_{max}-\zeta_{min})\cos^2(\theta)\Big]\Big)d\theta \\
&\qquad (\text{as}\sin^2(\theta)+\cos^2(\theta)=1) \\
&= \frac{a_4(Q)}{2pi}\exp\Big(\frac{1}{4}\zeta_{min}\Big)\exp\Big(\frac{1}{8}(\zeta_{max}-\zeta_{min})\Big) \\
&\quad \int_{\widehat{\theta}-2pi}^{\widehat{\theta}} \exp\Big(\frac{1}{8}(\zeta_{max}-\zeta_{min})\cos(2\theta)\Big)d\theta \\
&\qquad (\text{as } \cos^2(\theta)=(1+\cos(2\theta))/2) \\
&= \frac{a_4(Q)}{4pi}\exp\Big(\frac{1}{4}\zeta_{min}\Big)\exp\Big(\frac{1}{8}(\zeta_{max}-\zeta_{min})\Big) \\
x &\quad \int_{2(\widehat{\theta}-2pi)}^{2\widehat{\theta}} \exp\Big(\kappa(Q)\cos(\theta)\Big)d\theta \\
&\qquad \text{(where we have used the change of variable } \tilde{\theta}=2\theta) \\
&\qquad (\kappa(Q)\equiv\frac{1}{8}(\zeta_{max}-\zeta_{min})) \\
&= a_4(Q)\exp\Big(\frac{1}{4}\zeta_{min}\Big)\exp\Big(\frac{1}{8}(\zeta_{max}-\zeta_{min})\Big) \\
&\quad \frac{1}{2pi}\int_0^{2pi} \exp\Big(\kappa(Q)\cos(u)\Big)du \\
&\qquad \text{(where we have used the change of variable } u=(\widehat{\theta})-(\theta/2)
\end{aligned}
$$

Using the definition of the Von-Mises distribution and equation 3.5.18 in p. 36 of Mardia and Jupp (2000) it follows that:

$$
\begin{aligned}
f^*(S,T) &= a_4(Q)\exp\Big(\frac{1}{4}\zeta_{min}\Big)\exp\Big(\frac{1}{8}(\zeta_{max}-\zeta_{min})\Big)I_0\Big(\kappa(Q)\Big), \\
&= a_4(Q)\exp\Big(\frac{1}{8}\Big(\zeta_{max}+\zeta_{min}\Big)\Big)I_0\Big(\frac{1}{8}(\zeta_{max}-\zeta_{min})\Big)
\end{aligned}
$$

where $I_0(\cdot)$ is the modified Bessel function of the first kind, defined in Abramowitz and Stegun (1964), Section 9.6, p. 375 and

$$
a_4(Q) \propto (2pi)^{k/2}a_2(Q)2/(2^{k/2}\Gamma(k/2))\exp\Big(-\frac{1}{2}[S'S+T'T]\Big)
$$

Q.E.D.

PROOF OF RESULT 1: I now derive the WAP-similar test. From the definitions of ζ_{max} and ζ_{min}:

$$\frac{1}{8}\Big(\zeta_{max}+\zeta_{min}\Big)=\frac{1}{8}S'S+T'T, \quad \frac{1}{8}\Big(\zeta_{max}-\zeta_{min}\Big)=\frac{1}{8}\Big((S'S-T'T)^2+4(S'T)^2\Big)^{1/2}.$$

From Step 3 (aux) above it follows that the integrated likelihood for independent weights:

$$\phi\sim\mathcal{U}(\mathcal{S}^1) \quad \omega\sim\mathcal{U}(\mathcal{S}^k) \quad \rho\sim\sqrt{\chi^2_k}$$

is given by:

$$f^*(S,T)=cons_1\exp(-\frac{1}{2}[S'S+T'T])\exp\left(\frac{1}{8}\Big(S'S+T'T\Big)\right)I_0\left(\frac{1}{8}\Big((S'S-T'T)^2+4(S'T)^2\Big)^{1/2}\right)$$

Note that the denominator in the expression of the WAP-similar test is

$$f(S|T;\beta_0)=cons_2\exp\Big(-\frac{1}{2}S'S\Big).$$

Consequently:

$$z_{\text{WAP}}(S,T)=\frac{cons_1}{cons_2}\exp\Big(-\frac{1}{2}T'T\Big)\exp\Big(\frac{1}{8}[S'S+T'T]\Big)I_0\left(\frac{1}{8}\Big((S'S-T'T)^2+4(S'T)^2\Big)^{1/2}\right)$$

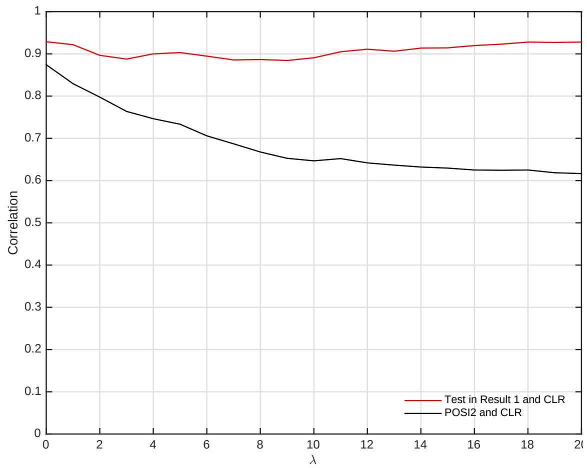
The quantile function $c(T,\alpha)$ is continuous in T and, therefore, measurable. So that WAP-similar test rejects if and only if the test statistic

$$S'S-T'T+8\ln\left[I_0\Big(\frac{1}{8}\Big((S'S-T'T)^2+4(S'T)^2\Big)^{1/2}\Big)\right]$$

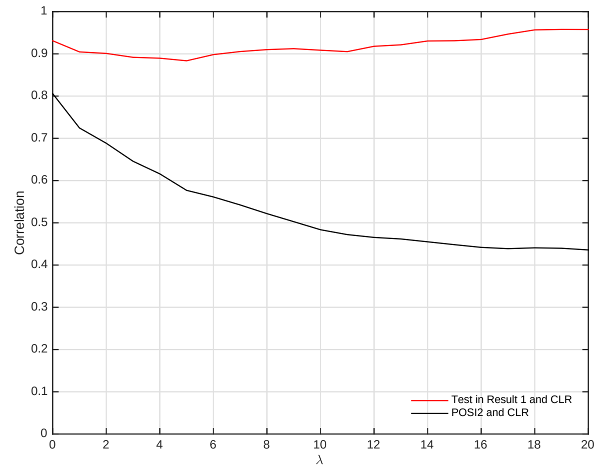
is larger than the critical value function $c^*(T,\alpha)$, defined as the $1-\alpha$ quantile (conditional on T) of the expression above under the distribution $S\sim\mathcal{N}_k(\mathbf{0},\mathbb{I}_k)$. This test is equivalent to the one presented in Result 1.

B.3. *Additional Figures*

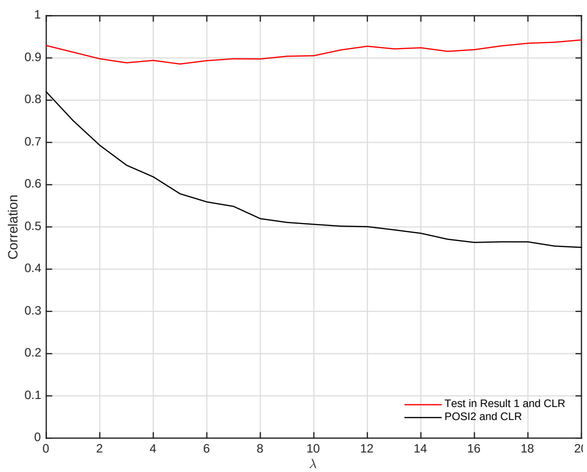
Figure 6: Correlation between the WAP-similar test and the CLR ($k = 4$)



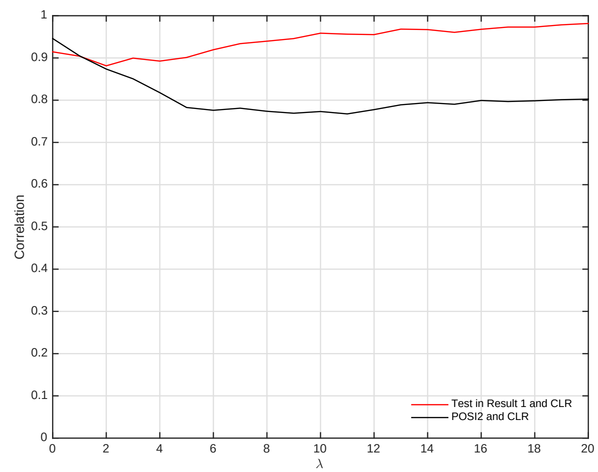
(a) $r = .2$



(b) $r = .4$



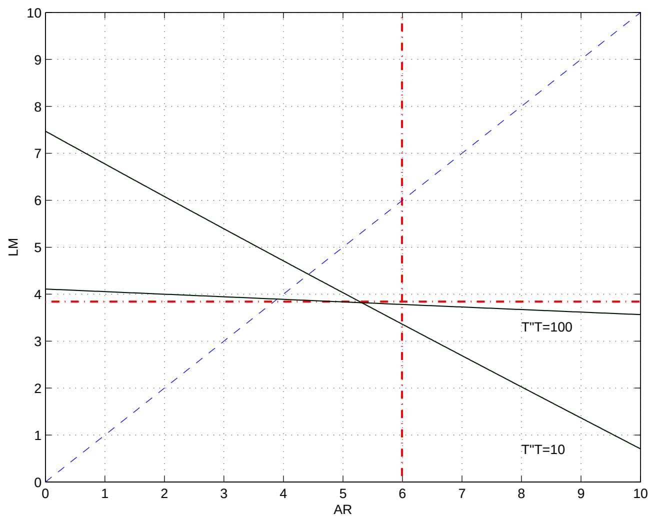
(c) $r = .6$



(d) $r = .8$

Description: Figure 6 reports the correlation (under the null hypothesis) between the WAP-similar test in Result 1 and both the CLR and the POSI2 in Andrews et al. (2006). The design under consideration is almost the same as in Figure 3. The null hypothesis is $\beta_0 = 0$. The matrix Ψ is assumed to have unit diagonal elements and correlation parameter $r \in \{.2, .4, .6, .8\}$. The matrix $\Phi = \mathbb{I}_k$. The number of instruments is $k = 4$. It has been argued that the power of the CLR is close to that of POSI2 and I include it for comparison. The POSI2 test requires to be evaluated a pair (β, λ). I consider the alternative $\beta = 2.1$ and $\lambda = 1$. The alternative for β is chosen to be close to $1/r = 2$. Even though the rate of Type I error of these tests does not depend on the concentration parameter, their correlation changes with λ. I use a uniform grid for $\lambda \in [0, 20]$ of 20 points. The script used to generate this figure is `CorrelationWAPvsCLR.m`.

Figure 7: 5% Conditional Critical Region
(AR,LM), $k = 2$



(Blue, dashed) Boundary of the sample space: AR $\geq$ LM, where AR$\equiv$ $S'S$ and LM$\equiv$ $(S'T)^2/(T'T)$. (Red, dot-dashed) 5% critical values for the AR and the LM statistics obtained as the upper 5% quantiles of the distributions χ^2_2 and χ^2_1, respectively.

The conditional critical region is the collection of (AR, LM) points at the right of the black (solid) lines (large AR and large LM). Each solid line traces the boundary of the rejection region of the WAP-similar test for a given value of $T'T \in \{10, 100\}$. The command `ezplot` in Matlab is used to graph the solution to the equation z(AR, LM, $T'T$)-$c(T'T;\alpha) = 0$.

B.4. *Asymptotic validity of the test in Result 1*

The test in Result 1 was derived under the assumption that the rotated reduced-form OLS estimators $(S_n', T_n')'$ have the exact distribution:

$$Q^n_{\beta,\pi,\Sigma} \equiv \mathcal{N}_{2k}\left(\begin{array}{c} ([b_0' \otimes \mathbb{I}_k)\Sigma(b_0 \otimes \mathbb{I}_k)]^{-1/2}(\beta-\beta_0)\sqrt{n}\pi \\ [(a_0' \otimes \mathbb{I}_k)\Sigma^{-1}(a_0 \otimes \mathbb{I}_k)]^{-1/2}(a_0' \otimes \mathbb{I}_k)\Sigma^{-1}(a \otimes \mathbb{I}_k)\sqrt{n}\pi \end{array}, \ \mathbb{I}_{2k} \right),$$

where Σ is of the form $\Psi \otimes \Phi$. In any finite sample, however, the law of $(S_n', T_n')'$ is a function of (β, π), the sample size, and the joint distribution between the instrumental variables and reduced-form residuals, denoted F. In fact, one can write:

$$\left(\begin{array}{c} ([b_0' \otimes \mathbb{I}_k)\widehat{\Sigma}(b_0 \otimes \mathbb{I}_k)]^{-1/2}(b_0' \otimes \mathbb{I}_k) \\ [(a_0' \otimes \mathbb{I}_k)\widehat{\Sigma}^{-1}(a_0 \otimes \mathbb{I}_k)]^{-1/2}(a_0' \otimes \mathbb{I}_k)\widehat{\Sigma}^{-1} \end{array} \right) \sqrt{n}\widehat{\gamma}_n \sim P^n_{\beta,\pi,F},$$

where $\widehat{\Sigma}$ is an estimator of the variance of $\sqrt{n}\widehat{\gamma}_n$. This variance depends on F and such dependence is denoted $\Sigma(F)$. The estimator $\widehat{\Sigma}$ need not have the Kronecker form, even when $\Sigma(F)$ does.

If one assumes that for n large enough the distributions P^n and Q^n are 'close' to each other (under the null), then one would expect the rate of Type I error computed under P^n to be close to that obtained under Q^n.

Preliminaries: I introduce some additional notation in order to establish Part 2 of Result 1.

1. *Bounded Lipschitz Distance*: Let $d_{\text{BL}}(P,Q) = \sup_{h \in \text{BL}_1} |\mathbb{E}_P[h(X)] - \mathbb{E}_Q[h(X)]|$ denote the Bounded Lipschitz distance between any pair of probability measures P and Q. For definitions and notation, see p. 73, Section 1.12 of Van der Vaart and Wellner (1996). Note also that the Bounded Lipschitz metric is equivalent to the 'β' metric between Borel probability measures defined in p. 394 of Dudley (2002).

2. *δ-Expansion of a set A:* For any $\delta > 0$ let A^δ denote the δ-expansion of the set $A \subseteq \mathbb{R}^m$. This is $A^\delta = \{y \in \mathbb{R}^m \mid d(x,y) \leq \delta \text{ for some } x \in A\}$.

3. *A bound on the distance between probability measures:* One can show that for any measurable set A and any $\delta > 0$:

$$\text{(B.11)} \qquad -Q((A^c)^\delta \backslash A^c) - \frac{1}{\delta} d_{\text{BL}_1}(P,Q) \leq P(A) - Q(A) \leq \frac{1}{\delta} d_{\text{BL}_1}(P,Q) + Q(A^\delta \backslash A),$$

where A^c is the complement of $A \subseteq \mathbb{R}^m$. I use the right-hand side of this inequality to establish the main result.

Assumption L0: Suppose that the class of distributions $\mathcal{F}$ is such that:

$$\lim_{n\to\infty} \sup_{(\pi,F) \in \mathbb{R}^k \times \mathcal{F}} d_{\text{BL}}\left(P^n_{\beta_0,\pi,F}, Q^n_{\beta_0,\pi,\Sigma(F)}\right) \to 0.$$

That is, the Bounded Lipschitz distance between the measures $P^n_{\beta,\pi,F}$ and $Q^n_{\beta,\pi,\Sigma(F)}$ coverges to zero as the sample size grows large (uniformly over π and F).

Asymptotic Validity of the Test in Result 1: If Assumption L0 holds and there are constants $0 < \underline{\lambda} < \overline{\lambda} < \infty$ such that the eigenvalues of $\Sigma(F)$ belong to an interval $[\underline{\lambda}, \overline{\lambda}]$ for any $F \in \mathcal{F}$, then:

$$\limsup_{n\to\infty} \sup_{(\pi,F) \in \mathbb{R}^k \times \mathcal{F}} P^n_{\beta_0,\pi,F}\left(z_{\text{WAP}}(S,T) - c_{\text{WAP}}(T,\alpha) > 0\right) \leq \alpha.$$

This means that the rate of Type I error of the test in Result 1 is uniformly controlled over $(\pi, \mathcal{F}) \in \mathbb{R}^k \times \mathcal{F}$.

Consider the test statistic

$$z(S,T) \equiv S'S - T'T + 8\ln\left(I_0\left[(1/8)\sqrt{(S'S-T'T)^2 + 4(S'T)^2}\right]\right)$$

and let $c(T;\alpha)$ denote its conditional critical value. I would like to show that if Assumption L0 holds over the class $\mathcal{F}$, then:

$$\limsup_{n\to\infty} \sup_{(\pi,F)\in\mathbb{R}^k\times\mathcal{F}} P^n_{\beta_0,\pi,F}\left(z(S,T) - c(T;\alpha) \geq 0\right) \leq \alpha. \tag{B.12}$$

I establish the asymptotic validity of the test in Result 1 in 6 steps:

Step 0: Define

$$A \equiv \{(s',t')' \in \mathbb{R}^{2k} \mid z(s,t) - c(t;\alpha) \geq 0\}.$$

Note immediately that Equation (B.11) implies that for any sample size n and any $(\pi, F) \in \mathbb{R}^k \times \mathcal{F}$:

$$\begin{aligned} P^n_{\beta_0,\pi,\mathcal{F}}(A) &\leq Q^n_{\beta_0,\pi,\Sigma(F)}(A) + \frac{1}{\delta} d_{\mathrm{BL}_1}(P^n_{\beta_0,\pi,F}, Q^n_{\beta_0,\pi,\Sigma(F)}) + Q^n_{\beta_0,\pi,\Sigma(F)}(A^\delta \backslash A), \\ &= \alpha + \frac{1}{\delta} d_{\mathrm{BL}_1}(P^n_{\beta_0,\pi,F}, Q^n_{\beta_0,\pi,\Sigma(F)}) + Q^n_{\beta_0,\pi,\Sigma(F)}(A^\delta \backslash A). \end{aligned}$$

where the last equality follows by the definition of the conditional critical value $c(T;\alpha)$. Thus, in order to establish (B.12), I need to show that

$$\frac{1}{\delta} d_{\mathrm{BL}_1}(P^n_{\beta_0,\pi,F}, Q^n_{\beta_0,\pi,\Sigma(F)}) + Q^n_{\beta_0,\pi,\Sigma(F)}(A^\delta \backslash A)$$

can be made arbitrary small, uniformly over the values of (π, F). By the weak convergence assumption in Part 2 of Result 1, for any fixed δ there is $M_\epsilon(\delta) \in \mathbb{N}$ such that whenever $n \geq M_\epsilon(\delta)$ the term $\delta^{-1} d_{\mathrm{BL}_1}(P^n_{\beta_0,\pi,F}, Q^n_{\beta_0,\pi,\Sigma(F)})$ can be made smaller than ϵ. Thus, I only need to establish the following result.

Goal: For every $\epsilon > 0$ there is δ_ϵ and N_ϵ such that for all $n \geq N_\epsilon$

$$\sup_{(\pi,F)\in\mathbb{R}^k\times F} Q^n_{\beta_0,\pi,\Sigma(F)}(A^{\delta_\epsilon} \backslash A) \leq \epsilon.$$

The proof of this result requires a series of intermediate steps. I exploit the fact that the test statistic $z(s,t)$ satisfies a Lipschitz condition whenever (s,t) is restricted to an appropriate set.

Step 1: (A bound on $Q^n_{\beta_0,\pi,\Sigma(F)}(A^\delta \backslash A)$): Define the sets

$$B(\underline{b}_1, \overline{b}_2) \equiv \{(s',t')' \in \mathbb{R}^{2k} \mid s's \in [\underline{b}_1, \overline{b}_2]\ \}, \tag{B.13}$$

$$C(\underline{c}_1, \overline{c}_2) \equiv \{(s',t')' \in \mathbb{R}^{2k} \mid t't \in [\underline{c}_1, \overline{c}_2]\ \}, \tag{B.14}$$

$$D(\underline{d}_1, \overline{d}_2) \equiv \{(s',t')' \in \mathbb{R}^{2k} \mid (s't)^2/t't \in [\underline{d}_1, \overline{d}_2]\ \}, \tag{B.15}$$

where $\underline{b}_1, \overline{b}_2, \underline{c}_1, \overline{c}_2, \underline{d}_1, \overline{d}_2$ are positive, finite constants. I want to study the behavior of $A^\delta \backslash A$ inside and outside the sets defined above. Note that for any n, π, F and δ:

$$
\begin{aligned}
Q^n_{\beta_0,\pi,\Sigma(F)}(A^\delta \backslash A) &= Q^n_{\beta_0,\pi,\Sigma(F)}(A^\delta \backslash A \cap [B(\underline{b}_1,\overline{b}_1)\cap C(\underline{c}_1,\overline{c}_2)\cap D(\underline{d}_1,\overline{d}_2)]) \\
&+ Q^n_{\beta_0,\pi,\Sigma(F)}(A^\delta \backslash A \cap [B(\underline{b}_1,\overline{b}_1)\cap C(\underline{c}_1,\overline{c}_2)\cap D(\underline{d}_1,\overline{d}_2)]^c) \\
&\quad \text{(by the additivity property of probability measures)} \\
&\leq Q^n_{\beta_0,\pi,\Sigma(F)}(A^\delta \backslash A \cap [B(\underline{b}_1,\overline{b}_1)\cap C(\underline{c}_1,\overline{c}_2)\cap D(\underline{d}_1,\overline{d}_2)]) \\
&+ Q^n_{\beta_0,\pi,\Sigma(F)}(A^\delta \backslash A \cap B^c(\underline{b}_1,\overline{b}_1)) + Q^n_{\beta_0,\pi\Sigma(F)}(A^\delta \backslash A \cap C^c(\underline{c}_1,\overline{c}_2)) \\
&+ Q^n_{\beta_0,\pi,\Sigma(F)}(A^\delta \backslash A \cap D(\underline{d}_1,\overline{d}_2)) \\
&\quad \text{(where I have used Boole's inequality)} \\
&\leq Q^n_{\beta_0,\pi,\Sigma(F)}(A^\delta \backslash A \cap [B(\underline{b}_1,\overline{b}_1)\cap C(\underline{c}_1,\overline{c}_2)\cap D(\underline{d}_1,\overline{d}_2)]) \\
&+ Q^n_{\beta_0,\pi,\Sigma(F)}(B^c(\underline{b}_1,\overline{b}_1)) + Q^n_{\beta_0,\pi\Sigma(F)}(D^c(\underline{d}_1,\overline{d}_2)) \\
&+ Q^n_{\beta_0,\pi\Sigma(F)}(A^\delta \backslash A \cap C^c(\underline{c}_1,\overline{c}_2)) \\
&\quad \text{(by the monotonicity of probability measures)} \\
&\leq Q^n_{\beta_0,\pi,\Sigma(F)}(A^\delta \backslash A \cap [B(\underline{b}_1,\overline{b}_1)\cap C(\underline{c}_1,\overline{c}_2)\cap D(\underline{d}_1,\overline{d}_2)]) \\
&+ \mathbb{P}(\chi^2_k \notin (\underline{b}_1,\overline{b}_2)) + \mathbb{P}(\chi^2_1 \notin (\overline{d}_1,\overline{d}_2)) \\
&+ Q^n_{\beta_0,\pi\Sigma(F)}(A^\delta \backslash A \cap C^c(\underline{c}_1,\overline{c}_2)),
\end{aligned}
$$

where the second to last line has uses the fact that under any probability measure $Q^n_{\beta_0,\pi,\Sigma(F)}$:[15]

$$S_n'S_n \overset{Q^n_{\beta_0,\pi,F}}{\sim} \chi^2_k \text{ and } (S_n'T_n)^2/T_n'T_n \overset{Q^n_{\beta_0,\pi,F}}{\sim} \chi^2_1.$$

<u>Main conclusion of Step 1:</u> I have shown that for any $\delta > 0$ and any positive finite constants $\underline{b}_1, \overline{b}_2, \underline{c}_1, \overline{c}_2, \underline{d}_1, \overline{d}_2$:

$$
\begin{aligned}
Q^n_{\beta_0,\pi,\Sigma(F)}(A^\delta \backslash A) &\leq Q^n_{\beta_0,\pi,\Sigma(F)}(A^\delta \backslash A \cap [B(\underline{b}_1,\overline{b}_1)\cap C(\underline{c}_1,\overline{c}_2)\cap D(\underline{d}_1,\overline{d}_2)]) \\
&+ \mathbb{P}(\chi^2_k \notin (\underline{b}_1,\overline{b}_2)) + \mathbb{P}(\chi^2_1 \notin (\overline{d}_1,\overline{d}_2)) \\
&+ Q^n_{\beta_0,\pi\Sigma(F)}(A^\delta \backslash A \cap C^c(\underline{c}_1,\overline{c}_2)).
\end{aligned} \tag{B.16}
$$

I now argue that for an appropriate selection of constants, the test statistic $z(s,t)$ and its critical value $c(t;\alpha)$ satisfy a Lipschitz condition when restricted to the set $[B(\underline{b}_1,\overline{b}_1)\cap C(\underline{c}_1,\overline{c}_2)\cap D(\underline{d}_1,\overline{d}_2)]$.

Step 2—Part a): (Lipschitz property of $z(s,t)$): I show that there exists a constant M_1—that only depends on the sets B, C, D—such that for any

$$(s_0',t_0')', (s_1',t_1')' \in \mathcal{K} \equiv [B(\underline{b}_1,\overline{b}_1)\cap C(\underline{c}_1,\overline{c}_2)\cap D(\underline{d}_1,\overline{d}_2)],$$

then:

$$|z(s_0,t_0) - z(s_1,t_1)| < M_1 ||(s_0',t_0') - (s_1',t_1')||.$$

To verify the Lipschitz property on $[B(\underline{b}_1,\overline{b}_1)\cap C(\underline{c}_1(\epsilon),\overline{c}_2(\epsilon))\cap D(\underline{d}_1,\overline{d}_2)]$, it is sufficient to show that, over this set, the derivative of $z(s,t)$ is continuous in its arguments. This observation, together with the fact that $[B(\underline{b}_1,\overline{b}_1)\cap C(\underline{c}_1(\epsilon),\overline{c}_2(\epsilon))\cap D(\underline{d}_1,\overline{d}_2)]$ is compact gives the desired result. Note that the partial derivative of $z(s,t)$ with respect to s is given by:

$$z_s(s,t) = 2S + \frac{8I_1\left((1/8)\sqrt{(s's-t't)^2 + 4t't(s't)^2/t't}\right)}{I_0\left((1/8)\sqrt{(s's-t't)^2 + 4t't(s't)^2/t't}\right)} \frac{1}{8} \frac{4(s's-t't)s + 8t't((s't)/t't)s}{2\sqrt{(s's-t't)^2 + 4t't((s't)^2/t't)}} \tag{B.17}$$

[15]I also use the fact that $\Sigma(F)$ is invertible for any element $F \in \mathcal{F}$.

$$z_t(s,t) = 2t + \frac{8I_1\left((1/8)\sqrt{(s's-t't)^2+4t't(s't)^2/t't}\right)}{I_0\left((1/8)\sqrt{(s's-t't)^2+4t't(s't)^2/t't}\right)}\frac{1}{8}\frac{4(s's-t't)t+8t't((s't)/t't)t}{2\sqrt{(s's-t't)^2+4t't((s't)^2/t't)}} \tag{B.18}$$

where $I_v(\cdot)$ is the modified Bessel function of the first kind defined in Section 9.6, p. 374 of Abramowitz and Stegun (1964). The formulae above use the fact that the derivative of the modified Bessel function of the first kind of order 0, I_0, is the modified Bessel function of order 1, I_1; see formula 9.6.27 in p. 376 of Abramowitz and Stegun (1964). The continuity of the derivatives and the fact that:

$$\sqrt{(s's-t't)^2+4t't(s't)^2/t't}$$

is bounded away from zero over the set

$$[B(\underline{b}_1,\overline{b}_1)\cap C(\underline{c}_1,\overline{c}_2)\cap D(\underline{d}_1,\overline{d}_2)]$$

implies that the Lipschitz condition holds.

Step 2—Part b): (Lipschitz property of $c(t;\alpha)$): Part a) showed that for any selection of constants (b,c,d) the test statistic $z(s,t)$ satisfies the Lipschitz condition when restricted to $\mathcal{K}$. I now introduce a parameter γ and show that for any given $\gamma>0$, $\epsilon>0$ and any pair of constants $\underline{c}_1$, $\overline{c}_2$, one can find $\underline{b}_1^*, \overline{b}_2^*$, $\underline{d}_1^*$, $\overline{d}_2^*$, and M_2—that depend on $\underline{c}_1$, $\overline{c}_2$, γ and ϵ—such that for any t_0,t_1 satisfying:

$$(s_0,t_0),(s_1,t_1)\in\mathcal{K}^*\equiv[B(\underline{b}_1^*,\overline{b}_1^*)\cap C(\underline{c}_1,\overline{c}_2)\cap D(\underline{d}_1^*,\overline{d}_2^*)], \quad \text{for some } s_0,s_1\in\mathbb{R}^k$$

the critical value function satisfies a Lipschitz-type condition:

$$|c(t_0;\alpha)-c(t_1;\alpha)|<M_2||t_0-t_1||+\gamma/2,$$

and:

$$\mathbb{P}(\chi_k^2\notin(\underline{b}_1^*,\overline{b}_1^*)+\mathbb{P}(\chi_1^2\notin(\underline{d}_1^*,\overline{d}_2^*))<\epsilon/3.$$

The parameter γ controls how close to is the conditional critical value to satisfy the Lipschitz condition.

To show this, note that for any constant $z\in\mathbb{R}$ and $(\pi,\mathcal{F})\in\mathbb{R}^k\times\mathcal{F}$:

$$\begin{aligned}
Q^n_{\beta_0,\pi,\Sigma(F)}\left(z(s,t_0)\le z \mid t=t_0\right) &= \mathbb{P}\left(z(S,t_0)\le z\right),\ S\sim\mathcal{N}(0,\mathbb{I}_k)\\
&= \mathbb{P}\left(z(S,t_0)-z(S,t_1)+z(S,t_1)\le z\right)\\
&= \mathbb{P}\left(z(S,t_1)\le z-\left(z(S,t_0)-z(S,t_1)\right) \text{ and } (S,t_0),(S,t_1)\in\mathcal{K}\right)\\
&+ \mathbb{P}\left(z(S,t_1)\le z-\left(z(S,t_0)-z(S,t_1)\right) \text{ and } (S,t_0),(S,t_1)\in\mathcal{K}^c\right).
\end{aligned}$$

where $\mathcal{K}\equiv[B(\underline{b}_1,\overline{b}_1)\cap C(\underline{c}_1,\overline{c}_2)\cap D(\underline{d}_1,\overline{d}_2)]$. Since $z(s,t)$ satisfies the Lipschitz condition in $\mathcal{K}$ with constant $M_1(\mathcal{K})$ it follows that:

$$\begin{aligned}
\mathbb{P}\Big(z(S,t_1)\le z-M_1(\mathcal{K})||t_0-t_1||\Big) &\le \mathbb{P}\Big(z(S,t_1)\le z-M_1(\mathcal{K})||t_0-t_1|| \text{ and } (S,t_0),(S,t_1)\in\mathcal{K}\Big)\\
&+ \mathbb{P}\Big(z(S,t_1)\le z-M_1(\mathcal{K})||t_0-t_1|| \text{ and } (S,t_0),(S,t_1)\in\mathcal{K}^c\Big)\\
&\le \mathbb{P}\Big(z(S,t_1)\le z-\left(z(S,t_0)-z(S,t_1)\right) \text{ and } (S,t_0),(S,t_1)\in\mathcal{K}\Big)\\
&+ \mathbb{P}((S,t_0),(S,t_1)\in\mathcal{K}^c),
\end{aligned}$$

which implies that:

$$\mathbb{P}\Big(z(S,t_1)\le z-M_1(\mathcal{K})||t_0-t_1||\Big)-\mathbb{P}((S,t_0),(S,t_1)\in\mathcal{K}^c)$$

is less than or equal to

$$\mathbb{P}\Big(z(S,t_1) \leq z - (z(S,t_0) - z(S,t_1)) \text{ and } (S,t_0),(S,t_1) \in \mathcal{K}\Big).$$

Note also that:

$$\mathbb{P}\Big(z(S,t_1) \leq z - (z(S,t_0) - z(S,t_1)) \text{ and } (S,t_0),(S,t_1) \in \mathcal{K}\Big) \leq \mathbb{P}\Big(z(S,t_1) \leq z + M_1(\mathcal{K})||t_0 - t_1||\Big).$$

Note now that for any $t \in \mathbb{R}^k$, the critical value function is continuous in α. Therefore, there exists a positive constant, $\eta_\gamma(\underline{c}_1, \overline{c}_2) > 0$, such that for any t such that $t't \in [\underline{c}_1, \overline{c}_2]$:

$$|c(t;\alpha + \eta_\gamma(\underline{c}_1,\overline{c}_2)) - c(t;\alpha)| \leq \gamma/2, \quad |c(t;\alpha - \eta_\gamma(\underline{c}_1,\overline{c}_2)) - c(t;\alpha)| \leq \gamma/2.$$

Since for any vectors $t_0, t_1 \neq \mathbf{0}_{k\times 1}$:

$$\mathbb{P}((S,t_0),(S,t_1) \in \mathcal{K}^c) \leq \mathbb{P}(\chi^2_k \notin (\underline{b}_1, \overline{b}_1) + \mathbb{P}(\chi^2_1 \notin (\underline{d}_1, \overline{d}_2)),$$

one can then choose $0 < \underline{b}_1^* \equiv \underline{b}_1(\eta_\gamma(\underline{c}_1,\overline{c}_2),\epsilon) < \overline{b}_1^* \equiv \overline{b}_1(\eta_\gamma(\underline{c}_1,\overline{c}_2),\epsilon) < \infty$, and $0 < \underline{d}_1^* \equiv \underline{d}_1(\eta_\gamma(\underline{c}_1,\overline{c}_2),\epsilon) < \overline{d}_2^* \equiv \overline{d}_2(\eta_\gamma(\underline{c}_1,\overline{c}_2),\epsilon) < \infty$ such that

$$\mathbb{P}(\chi^2_k \notin (\underline{b}_1^*, \overline{b}_1^*) + \mathbb{P}(\chi^2_1 \notin (\underline{d}_1^*, \overline{d}_2^*)) < \min\{\eta_\gamma(\underline{c}_1,\overline{c}_2), \epsilon/3\}$$

for any t_0, t_1. This implies that:

$$\text{(B.19)} \qquad \mathbb{P}\Big(z(S,t_1) \leq z - M_2(\underline{c}_1,\overline{c}_2,\gamma,\epsilon)||t_0 - t_1||\Big) - \eta_\gamma(\underline{c}_1,\overline{c}_2) \;\leq\; \mathbb{P}\left(z(S,t_0) \leq z\right)$$

$$\text{(B.20)} \qquad \mathbb{P}\left(z(S,t_0) \leq z\right) \leq \mathbb{P}\Big(z(S,t_1) \leq z + M_2(\underline{c}_1,\overline{c}_2,\gamma,\epsilon)||t_0 - t_1||\Big) + \eta_\gamma(\underline{c}_1,\overline{c}_2),$$

where

$$M_2(\underline{c}_1,\overline{c}_2,\gamma,\epsilon) \equiv M_1\Big(\underline{b}_1(\eta_\gamma(\underline{c}_1,\overline{c}_2),\epsilon), \overline{b}_1(\eta_\gamma(\underline{c}_1,\overline{c}_2),\epsilon), \underline{c}_1, \overline{c}_2, \underline{d}_1(\eta_\gamma(\underline{c}_1,\overline{c}_2),\epsilon), \overline{d}_2(\eta_\gamma(\underline{c}_1,\overline{c}_2),\epsilon)\Big).$$

For simplicity we write M_2 instead of $M_2(\underline{c}_1,\overline{c}_2,\gamma,\epsilon)$ whenever it is convenient.

Since (B.19) holds for any z, in particular it holds for $z = c(t_1;\alpha) + M_2||t_0 - t_1||$. Consequently:

$$\begin{aligned} \mathbb{P}\left(z(S,t_0) \leq c(t_1;\alpha) + M_2||t_0 - t_1||\right) &\geq \mathbb{P}\left(z(S,t_1) \leq c(t_1;\alpha)\right) - \eta_\gamma(\underline{c}_1,\overline{c}_2) \\ &= 1 - \alpha - \eta_\gamma(\underline{c}_1,\overline{c}_2). \end{aligned}$$

This implies that

$$\text{(B.21)} \qquad c(t_0;\alpha + \eta_\gamma(\underline{c}_1,\overline{c}_2)) \leq c(t_1;\alpha) + M_2||t_0 - t_1||.$$

Likewise, equation (B.20) holds for any z, in particular it holds for $z = c(t_1;\alpha) - M_2||t_0 - t_1||$. This implies that:

$$\begin{aligned} \mathbb{P}\left(z(S,t_0) \leq c(t_1;\alpha) - M_2||t_0 - t_1||\right) &\leq \mathbb{P}\left(z(S,t_1) \leq c(t_1;\alpha)\right) + \eta_\gamma(\underline{c}_1,\overline{c}_2) \\ &= (1 - \alpha) + \eta_\gamma(\underline{c}_1,\overline{c}_2). \end{aligned}$$

This implies that:

$$\text{(B.22)} \qquad c(t_0;\alpha - \eta_\gamma(\underline{c}_1,\overline{c}_2)) \geq c(t_1;\alpha) - M_2||t_0 - t_1||.$$

MAIN CONCLUSION OF STEP 2: Finally, (B.21)-(B.22) and the definition of $\eta_\gamma(\underline{c}_1,\overline{c}_2)$ imply that for any $\gamma > 0$, $\epsilon > 0$ and any pair of constants $\underline{c}_1$, $\overline{c}_2$, one can find $\underline{b}_1^*, \overline{b}_2^*$, $\underline{d}_1^*$, $\overline{d}_2^*$, and M_2—that depend on $\underline{c}_1$, $\overline{c}_2$, γ and ϵ—such that for any:

$$(s_0,t_0),(s_1,t_1) \in \mathcal{K}^* \equiv [B(\underline{b}_1^*,\overline{b}_1^*) \cap C(\underline{c}_1,\overline{c}_2) \cap D(\underline{d}_1^*,\overline{d}_2^*)],$$

the critical value function satisfies the Lipschitz-type condition:

$$|c(t_0;\alpha) - c(t_1;\alpha)| < M_2||t_0 - t_1|| + \gamma/2.$$

and:

$$\mathbb{P}(\chi_k^2 \notin (\underline{b}_1^*,\overline{b}_1^*) + \mathbb{P}(\chi_1^2 \notin (\underline{d}_1^*,\overline{d}_2^*)) < \epsilon/3.$$

Step 3: (Exploiting the Lipschitz property to manipulate $A^\delta \backslash A$) The constants in Step 2 depend on $\gamma > 0, \epsilon > 0$ and $\underline{c}_1, \overline{c}_2$. This step fixes $\gamma > 0$ and shows how to choose an appropriate enlargement of the set A as a function of γ. The Lipschitz condition established at the end of Step 2 allows for a convenient upper 'bound' on the set:

$$A^{\delta(\gamma)} \backslash A \ \cap [B(\underline{b}_1^*,\overline{b}_1^*) \cap C(\underline{c}_1,\overline{c}_2) \cap D(\underline{d}_1^*,\overline{d}_2^*)]. \tag{B.23}$$

In particular, I show that for any $\gamma > 0$, there exists $\delta(\gamma)$ such that:

$$(s',t')' \in A^{\delta(\gamma)} \backslash A \ \cap [B(\underline{b}_1^*,\overline{b}_1^*) \cap C(\underline{c}_1,\overline{c}_2) \cap D(\underline{d}_1^*,\overline{d}_2^*)]$$

implies that:

$$-\gamma \leq z(s,t) - c(t;\alpha) \leq 0.$$

This inclusion relation is convenient as it allows for the selection of the auxiliary parameter γ to make the probability of the set (B.23) uniformly small over (π, F).

To establish the desired result, note that $x \equiv (s',t')' \in A^\delta \backslash A$ implies that:

$$z(s,t) - c(t;\alpha) < 0, \ (\text{as } x \equiv (s',t')' \notin A),$$

and also that, for any δ, there exists $x_0(\delta) \equiv (s'_{0,\delta}, t'_{0,\delta})' \in A$ such that

$$d(x, x_0(\delta)) \leq \delta.$$

Since the functions $s's$, $t't$, $(s't)^2/(t't)$ defining the set:

$$\mathcal{K}^* \equiv [B(\underline{b}_1^*,\overline{b}_1^*) \cap C(\underline{c}_1,\overline{c}_2) \cap D(\underline{d}_1^*,\overline{d}_2^*)],$$

are Lipschitz continuous when restricted to $\mathcal{K}^*$, there exists δ^* small enough for which the corresponding $x_0(\delta^*)$ belongs to the set $\mathcal{K}^*$. In this case we have that:

$$\begin{aligned}
||(z(s,t) - c(t;\alpha)) - (z(s_{0,\delta^*}, t_{0,\delta^*}) - c(t_0;\alpha))|| \ &\leq \ ||z(s,t) - z(s_{0,\delta^*}, t_{0,\delta^*})|| + ||c(t;\alpha) - c(t_{0,\delta^*};\alpha)||, \\
&\leq \ (M_1(\mathcal{K}^*) + M_2(\underline{c}_1,\overline{c}_2,\gamma))d(x, x_0(\delta^*)) + \gamma/2, \\
& \ \ (\text{where I have used Step 2 part a) and b)}), \\
&\leq \ (M_1 + M_2)\delta^* + \gamma/2
\end{aligned}$$

Since $x \notin A$ and $x_0(\delta^*) \in A$ implies that

$$0 \geq (z(s,t) - c(t;\alpha)) \geq (z(s,t) - c(t;\alpha)) - (z(s_{0,\delta^*}, t_{0,\delta^*}) - c(t_0;\alpha)) \geq -(M_1 + M_2)\delta^* - \gamma/2$$

<u>Main conclusion of Step 3</u>: Taking $\delta(\gamma) \equiv \min\{\delta^*, \frac{\gamma}{2(M_1+M_2)}\}$ it follows that

$$(s',t')' \in A^{\delta(\gamma)} \backslash A \ \cap [B(\underline{b}_1^*,\overline{b}_1^*) \cap C(\underline{c}_1,\overline{c}_2) \cap D(\underline{d}_1^*,\overline{d}_2^*)]$$

implies that:

$$-\gamma \leq z(s,t) - c(t;\alpha) \leq 0.$$

I now exploit this relation to show that one can choose γ to guarantee that

$$\sup_{\pi, F \in \mathbb{R}^k \times \mathcal{F}} Q^n_{\beta_0,\pi,\mathcal{F}}(A^{\delta(\gamma)} \backslash A \, \cap [B(\underline{b}^*_1, \overline{b}^*_1) \cap C(\underline{c}_1, \overline{c}_2) \cap D(\underline{d}^*_1, \overline{d}^*_2)])$$

can be made arbitrarily small.

Step 4: (Choosing γ as a function of ϵ) Remember that equation (B.16) in Step 1 established that for any $\delta > 0$ and any constants $\underline{b}_1, \overline{b}_1, \underline{c}_1, \overline{c}_1, \underline{d}_1, \overline{d}_1$:

$$\begin{aligned} Q^n_{\beta_0,\pi,\Sigma(F)}(A^\delta \backslash A) \quad &\leq \quad Q^n_{\beta_0,\pi,\Sigma(F)}(A^\delta \backslash A \, \cap [B(\underline{b}_1, \overline{b}_1) \cap C(\underline{c}_1, \overline{c}_2) \cap D(\underline{d}_1, \overline{d}_2)]) \\ &+ \quad \mathbb{P}(\chi^2_k \notin (\underline{b}_1, \overline{b}_2)) + \mathbb{P}(\chi^2_1 \notin (\overline{d}_1, \overline{d}_2)) \\ &+ \quad Q^n_{\beta_0,\pi\Sigma(F)}(A^\delta \backslash A \, \cap C^c(\underline{c}_1, \overline{c}_2)). \end{aligned}$$

Step 4 showed that for any $\gamma > 0$ there is way of selecting the enlargement parameter $\delta(\gamma) > 0$ and constants $\underline{b}^*_1, \overline{b}^*_2, \underline{d}^*_1, \overline{d}^*_2$—that depend on $\underline{c}_1$, $\overline{c}_2$ and γ—such tha the probabilityt:

$$Q^n_{\beta_0,\pi,\Sigma(F)}(A^{\delta(\gamma)} \backslash A \, \cap [B(\underline{b}^*_1, \overline{b}^*_1) \cap C(\underline{c}_1, \overline{c}_2) \cap D(\underline{d}^*_1, \overline{d}^*_2)])$$

is less than or equal to

$$\text{(B.24)} \qquad Q^n_{\beta_0,\pi,\Sigma(F)}(-\gamma \leq z(s,t) - c(t;\alpha) \leq 0 \cap [B(\underline{b}^*_1, \overline{b}^*_1) \cap C(\underline{c}_1, \overline{c}_2) \cap D(\underline{d}^*_1, \overline{d}^*_2)])$$

I now show that there exists $\gamma_\epsilon > 0$ small enough such that:

$$Q^n_{\beta_0,\pi,\Sigma(F)}(-\gamma_\epsilon \leq z(s,t) - c(t;\alpha) \leq 0 \cap [B(\underline{b}^*_1, \overline{b}^*_1) \cap C(\underline{c}_1, \overline{c}_2) \cap D(\underline{d}^*_1, \overline{d}^*_2)]) < \epsilon/3$$

for any n, π, F.

To show this, define—for any t such that $t't \in [\underline{c}_1, \overline{c}_2]$—the function $\gamma_\epsilon(t)$ to satisfy:

$$\mathbb{P}_S(-\gamma_\epsilon(t) \leq g(S,t) \leq 0 \text{ and } S'S \in (\underline{b}^*_1, \overline{b}^*_1) \text{ and } \text{ and } (S't)^2/(t't) \in (\underline{d}^*_1, \overline{d}^*_1)), \quad S \sim \mathcal{N}(0, \mathbb{I}_k),$$

where $g(s,t;\alpha) \equiv z(s,t) - c(t;\alpha)$ and t is treated as fixed vector. Let

$$\gamma_\epsilon \equiv \inf_{\{t \mid t't \in [\underline{c}_1, \overline{c}_2]\}} \gamma_\epsilon(t)$$

and note that $\gamma_\epsilon > 0$ (otherwise, there will be a value t^* for which the distribution of $\mathbb{P}_S(g(S,t^*) = 0) > \epsilon/3)$. Note that for any n, π, F:

$$Q^n_{\beta_0,\pi,\Sigma(F)}(-\gamma_\epsilon \leq g(s,t;\alpha) \leq 0 \text{ and } s's \in (\underline{b}^*_1, \overline{b}^*_1) \text{ and } t't \in (\underline{c}_1, \overline{c}_2), \text{ and } (s't)^2/t't \in (\underline{d}^*_1, \overline{d}^*_1)),$$

is the same as:

$$\int_{t \in C(\underline{c}_1, \overline{c}_2)} \Big(Q^n_{\beta_0,\pi,\Sigma(F)}(-\gamma_\epsilon \leq g(s,t;\alpha) \leq 0 \text{ and } s's \in (\underline{b}^*_1, \overline{b}^*_1) \text{ and } \text{ and } (s't)^2/t't \in (\underline{d}^*_1, \overline{d}^*_1) \mid t) \Big) d\mathbb{P}^n_{\beta_0,\pi,F}(t),$$

where $\mathbb{P}^n_{\beta_0,\pi,F}$ is the marginal distribution that $Q^n_{\beta_0,\pi,F}$ induces over (t). Note that (B.25) equals:

$$\int_{t \in C(\underline{c}_1, \overline{c}_2)} \Big(\mathbb{P}_S(-\gamma_\epsilon \leq g(S,t) \leq 0 \text{ and } S'S \in (\underline{b}^*_1, \overline{b}^*_1) \text{ and } \text{ and } (S't)^2/(t't) \in (\underline{d}^*_1, \overline{d}^*_1)) \Big) d\mathbb{P}^n_{\beta_0,\pi,F}(t),$$

$s|t$ has distribution $\mathcal{N}_k(\mathbf{0}, \mathbb{I}_k)$ for any n, π, F. And this is smaller than or equal:

$$\int_{t\in C(\underline{c}_1,\overline{c}_2)}\Big(\mathbb{P}_S(-\gamma_\epsilon(t)\leq g(S,t)\leq 0 \text{ and } S'S\in(\underline{b}_1^*,\overline{b}_1^*) \text{ and } \text{ and } (S't)^2/(t't)\in(\underline{d}_1^*,\overline{d}_1^*))\Big)d\mathbb{P}^n_{\beta_0,\pi,F}(t),$$
$$= \mathbb{P}^n_{\beta_0,\pi,F}(t\in C(\underline{c}_1,\overline{c}_2))\frac{\epsilon}{3}<\frac{\epsilon}{3}, \text{ (by definition of } \gamma_\epsilon(t)).$$

<u>Main conclusion of Step 4</u>: This means that for any $\epsilon > 0$ there exists $\gamma_\epsilon > 0$ small enough such that:

$$Q^n_{\beta_0,\pi,\Sigma(F)}(-\gamma_\epsilon\leq z(s,t)-c(t;\alpha)\leq 0\cap[B(\underline{b}_1^*,\overline{b}_1^*)\cap C(\underline{c}_1,\overline{c}_2)\cap D(\underline{d}_1^*,\overline{d}_2^*)])<\epsilon/3$$

for any n,π,F.

Step 5 (choosing $\underline{c}_1$ and $\underline{c}_2$): Step 1 through Step 4 have shown that for any $\epsilon > 0$ there is a constant $\delta_\epsilon \equiv \delta(\gamma_\epsilon)$ and constants $\underline{b}_1^*, \overline{b}_2^*$, $\underline{d}_1^*$, $\overline{d}_2^*$—that depend on $\underline{c}_1$, $\overline{c}_2$ and ϵ such that for any n,π,F:

$$\begin{aligned}
Q^n_{\beta_0,\pi,\Sigma(F)}(A^{\delta_\epsilon}\backslash A) &\leq Q^n_{\beta_0,\pi,\Sigma(F)}(A^{\delta_\epsilon}\backslash A\cap[B(\underline{b}_1^*,\overline{b}_1^*)\cap C(\underline{c}_1,\overline{c}_2)\cap D(\underline{d}_1^*,\overline{d}_2^*)])\\
&+ \mathbb{P}(\chi^2_k\notin(\underline{b}_1^*,\overline{b}_2^*))+\mathbb{P}(\chi^2_1\notin(\overline{d}_1^*,\overline{d}_2^*))\\
&+ Q^n_{\beta_0,\pi,\Sigma(F)}(A^{\delta_\epsilon}\backslash A\cap C^c(\underline{c}_1,\overline{c}_2)).\\
&\leq \frac{2\epsilon}{3}+Q^n_{\beta_0,\pi,\Sigma(F)}(A^{\delta}\backslash A\cap C^c(\underline{c}_1,\overline{c}_2)).
\end{aligned}$$

This means that:

$$\sup_{(\pi,F)\in\mathbb{R}^k\times\mathcal{F}} Q^n_{\beta_0,\pi,\Sigma(F)}(A^{\delta_\epsilon}\backslash A)\leq\frac{2\epsilon}{3}+\sup_{(\pi,F)\in\mathbb{R}^k\times\mathcal{F}} Q^n_{\beta_0,\pi,\Sigma(F)}(A^{\delta_\epsilon}\backslash A\cap C^c(\underline{c}_1,\overline{c}_2)).$$

Thus, I only need to show is that $\underline{c}_1$ and $\overline{c}_2$ can be chosen to make the second term on the right of the inequality above smaller than $\epsilon/3$. Let λ^* be defined as:

$$\lambda^*\equiv\max_{F\in\mathcal{F}}(a_0'\otimes\mathbb{I}_k)\Sigma(F)^{-1}(a_0\otimes\mathbb{I}_k)$$

By assumption, there are constants $0<\underline{\lambda}<\overline{\lambda}<\infty$ such that $\underline{\lambda}<\lambda^*<\overline{\lambda}$. Fix $c^*\in\mathbb{R}^k$ and partition $\mathbb{R}^k$ as follows:

$$\{\pi\in\mathbb{R}^k\mid: n||\pi||^2\lambda^*\leq c^*\}\cup\{\pi\in\mathbb{R}^k\mid: n||\pi||^2\lambda^*>c^*\}\equiv\Pi_1^n(c^*)\cup\Pi_2^n(c^*).$$

Note that

$$\sup_{(\pi,F)\in\mathbb{R}^k\times\mathcal{F}} Q^n_{\beta_0,\pi\Sigma(F)}(A^{\delta_\epsilon}\backslash A\cap C^c(\underline{c}_1,\overline{c}_2))$$

is smaller than or equal than the sum of:

$$\sup_{(\pi,F)\in\Pi_1^n(c^*)\times\mathcal{F}} Q^n_{\beta_0,\pi,\Sigma(F)}(A^{\delta_\epsilon}\backslash A\cap C^c(\underline{c}_1,\overline{c}_2)),\tag{B.25}$$

and

$$\sup_{(\pi,F)\in\Pi_2^n(c^*)\times\mathcal{F}} Q^n_{\beta_0,\pi,\Sigma(F)}(A^{\delta_\epsilon}\backslash A\cap C^c(\underline{c}_1,\overline{c}_2))\tag{B.26}$$

Step 5—Part a): First, I bound the term (B.25). Let $\chi^2_k(c)$ denote a non-central chi-square with k degrees of freedom and centrality parameter c. Note that:

$$\begin{aligned}
\sup_{(\pi,F)\in\Pi_1^n(c^*)\times\mathcal{F}} Q^n_{\beta_0,\pi,\Sigma(F)}(A^{\delta_\epsilon}\backslash A \cap C^c(\underline{c}_1,\overline{c}_2)) &\leq \sup_{(\pi,F)\in\Pi_1^n(c^*)\times\mathcal{F}} Q^n_{\beta_0,\pi,\Sigma(F)}(C^c(\underline{c}_1,\overline{c}_2)),\\
&= \sup_{(\pi,F)\in\Pi_1^n(c^*)\times\mathcal{F}} Q^n_{\beta_0,\pi,\Sigma(F)}(t't \notin (\underline{c}_1,\overline{c}_2)),\\
&= \sup_{(\pi,F)\in\Pi_1^n(c^*)\times\mathcal{F}} \mathbb{P}\Big(\chi^2_k(n||\pi||^2\lambda(F)) \notin (\underline{c}_1,\overline{c}_2)\Big),\\
&\quad (\text{where } \lambda(F) \equiv (a_0'\otimes\mathbb{I}_k)\Sigma(F)^{-1}(a_0\otimes\mathbb{I}_k)).
\end{aligned}$$

Therefore, one can choose constants $\underline{c}_1^*, \overline{c}_2^*$ that depend on c^* and ϵ (but do not depend on the sample size) such that for any $\underline{c}_1 < \underline{c}_1^*$ and $\overline{c}_2 > \overline{c}_2^*$:

$$\sup_{(\pi,F)\in\Pi_1^n(c^*)\times\mathcal{F}} Q^n_{\beta_0,\pi,\Sigma(F)}(A^{\delta_\epsilon}\backslash A \cap C^c(\underline{c}_1,\overline{c}_2)) < \frac{\epsilon}{6}.$$

Step 5—Part b): Now, I bound the term (B.26). To do this, choose $\overline{e}$ to satisfy

$$\mathbb{P}(\chi^2_k > \overline{e}) < \frac{\epsilon}{12}.$$

Since this selection of $\overline{e}$ implies that

$$\sup_{(\pi,F)\in\Pi_2^n(c^*)\times\mathcal{F}} Q^n_{\beta_0,\pi,\Sigma(F)}(A^{\delta_\epsilon}\backslash A \ \cap\ s's > \overline{e} \ \cap\ C^c(\underline{c}_1,\overline{c}_2)) < \frac{\epsilon}{12},$$

it is sufficient to show that there is $\overline{c}_2$ large enough such that:

$$\sup_{(\pi,F)\in\Pi_2^n(c^*)\times\mathcal{F}} Q^n_{\beta_0,\pi,\Sigma(F)}(A^{\delta_\epsilon}\backslash A \ \cap\ s's \leq \overline{e} \ \cap\ C^c(\underline{c}_1,\overline{c}_2)) < \frac{\epsilon}{12}$$

The key to establish this result is to show that the for t't large enough and $s's$ in a compact set, the test statistic $z(s,t)$ defined in Result 1 is close to the statistic $(s't)^2/(t't)$.

Step 5—Part c): Let $\mathbf{o}(t't)$ denote the function:

$$\mathbf{o}(t't) \equiv \Big(((s's/t't)-1)^2 + 4(s't)^2/(t't)^2\Big)^{1/2} - 1.$$

Note that:

$$\begin{aligned}
& 8\ln\Big[I_0\Big((t't/8)(1+\mathbf{o}(t't))\Big)\Big],\\
=\ & 8\ln\Big[\frac{e^{(t't/8)(1+\mathbf{o}(t't))}}{\sqrt{2pi((t't/8)(1+\mathbf{o}(t't)))}}\Big(1+O\Big(\frac{1}{(t't/8)(1+\mathbf{o}(t't))}\Big)\Big)\Big],\\
& (\text{where I have used the asymptotic approximation}\\
& \text{for } I_0(z) \text{ in p. 435 of Olver (1997) and the definition}\\
& \text{of } \sim \text{ in p. 4 of the same book})\\
=\ & 8\ln\Big[\frac{e^{(t't/8)(1+\mathbf{o}(t't))}}{\sqrt{2pi((t't/8)(1+\mathbf{o}(t't)))}}\Big]
\end{aligned}$$

$$
\begin{aligned}
&+ \quad 8\ln\Big[\Big(1+O\Big(\frac{1}{(t't/8)(1+\mathbf{o}(t't))}\Big)\Big)\Big],\\
&= \quad t't(1+\mathbf{o}(t't)) - 4\ln(2pi) - 4\ln(t't/8) - 4\ln(1+\mathbf{o}(t't))\\
&+ \quad 8\ln\Big[\Big(1+O\Big(\frac{1}{(t't/8)(1+\mathbf{o}(t't))}\Big)\Big)\Big].
\end{aligned}
$$

Therefore, $z_{\text{WAP}}(s,t)$ in Result 1:

$$(s's-t't) + 8\ln\Big[\, I_0\Big(\frac{1}{8}\Big[(s's-t't)^2+4(s't)^2)\Big]^{1/2}\Big)\,\Big] + 4\ln(2pi) + 4\ln((1/8)t't)$$

can be written in terms of the Conditional Likelihood Ratio statistic (CLR) as follows:

$$
\begin{aligned}
z_{\text{WAP}}(s,t) \quad &\equiv \quad (s's-t't) + t't(1+\mathbf{o}(t't)) - 4\ln(1+\mathbf{o}(t't))\\
&+ \quad 8\ln\Big[\Big(1+O\Big(\frac{1}{(t't/8)(1+\mathbf{o}(t't))}\Big)\Big)\Big],\\
&= \quad 2\text{CLR}(s,t) - 4\ln(1+\mathbf{o}(t't))\\
&+ \quad 8\ln\Big[\Big(1+O\Big(\frac{1}{(t't/8)(1+\mathbf{o}(t't))}\Big)\Big)\Big],\\
&\quad\quad \text{(where we have used the fact that } t't(1+\boldsymbol{o}(t't))\\
&\quad\quad \text{equals } [(s's-t't)^2+(s't)]^{1/2}\ ).
\end{aligned}
$$

It is well-known for large values of $t't$ and for values of $s's$ in a compact set the CLR can be approximated by the LM statistic ($\equiv s't/t't$) uniformly over the values of s. Choose $\zeta^* > 0$ to satisfy:

$$\mathbb{P}(-\eta^* \le N(0,1) \le \eta^*) = \frac{\epsilon}{24}.$$

Therefore, using the same argument as in part b) of step 2 one can show for $\zeta^* > 0$ there is $\overline{c}_2^*$—that depends on ζ^*—such that uniformly over $s's \le \overline{e}$

$$|z_{\text{WAP}}(s,t) - c_{\text{WAP}}(t;\alpha) - 2LM(s,t) + 2\chi^2_{1,1-\alpha}| < \zeta^*,$$

where $\chi^2_{1,1-\alpha}$ is the 1-α quantile of a chi-squared random variable with one degree of freedom and $c_{\text{WAP}}(t;\alpha)$ is the conditonal critical value of $z_{\text{WAP}}(s,t)$.

Step 5—Part d): Note that

$$x \equiv (s,t) \in (A^{\delta_\epsilon} \backslash A \,\cap\, s's \leq \overline{e} \,\cap\, t't > \overline{c}_2^*),$$

implies that:

$$z_(s,t) - c(t;\alpha) < 0,$$

and also that there is $x_0(\delta_\epsilon) \equiv (s_0,t_0) \in A$ such that $z(s_0,t_0) - c(t;\alpha) > 0$ and $d(x, x_0(\delta_\epsilon)) < \delta_\epsilon$. Since the test based on the test statistic $z(s,t)$ with conditional critical value $c(t;\alpha)$ is equivalent to the test based on $z_{\text{WAP}}(s,t)$ and $c_{\text{WAP}}(t;\alpha)$, it follows that:

$$z_{\text{WAP}}(s,t) - c_{\text{WAP}}(t;\alpha) < 0, \text{ and } z_{\text{WAP}}(s_0,t_0) - c_{\text{WAP}}(t;\alpha) > 0.$$

Consequently:

$$\text{LM}(s,t) - \chi^2_{1,1-\alpha} < \zeta^*/2$$

and

$$\text{LM}(s_0,t_0) - \chi^2_{1,1-\alpha} > -\zeta^*/2.$$

Note that the LM statistic can be written as a function of (S, ω_t) where $\omega_t \equiv t/||t||$. Since the partial derivatives of $(s'\omega_t)^2$ are bounded whenever $s's \leq \overline{e}$, the LM statistic satisfies the Lipschitz condition when s's belongs to the desired domain. Let $M(\overline{e})$ denote the Lipschitz constant of the LM statistic. Since:

$$-d(x, x_0(\delta_\epsilon))M(\overline{e}) \leq \text{LM}(s,t) - \text{LM}(s_0,t_0) \leq M(\overline{e})d(x, x_0(\delta_\epsilon)),$$

then:

$$\begin{aligned}
-\delta_\epsilon M(\overline{e}) &\leq -d(x, x_0(\delta_\epsilon))M(\overline{e}) \\
&\leq \text{LM}(s,t) - \chi^2_{1,1-\alpha} + \chi^2_{1,1-\alpha} - \text{LM}(s_0,t_0), \\
&\leq \text{LM}(s,t) - \chi^2_{1,1-\alpha} + \zeta^*/2, \\
&\quad \text{(where I have used the fact that } \text{LM}(s_0,t_0) - \chi^2_{1,1-\alpha} > -\zeta^*/2), \\
&\leq \zeta^*, \\
&\quad \text{(where I have used the fact that } \text{LM}(s,t) - \chi^2_{1,1-\alpha} < \zeta^*/2).
\end{aligned}$$

One can further shrink δ_ϵ to satisfy $\delta_\epsilon M(\overline{e}) < -\zeta^*$. This means that:

$$\begin{aligned}
\sup_{(\pi,F)\in\Pi_2^n(c^*)\times\mathcal{F}} Q^n_{\beta_0,\pi,\Sigma(F)}(A^{\delta_\epsilon} \backslash A \,\cap\, s's \leq \overline{e} \,\cap\, t't > \overline{c}_2^*) &\leq \sup_{(\pi,F)\in\Pi_2^n(c^*)\times\mathcal{F}} Q^n_{\beta_0,\pi,\Sigma(F)}(-\eta^* \leq \text{LM}(s,t) \leq \eta^*) \\
&= \mathbb{P}(-\eta^* \leq N(0,1) \leq \eta^*) \\
&= \frac{\epsilon}{24}
\end{aligned}$$

Since:

$$\sup_{(\pi,F)\in\Pi_2^n(c^*)\times\mathcal{F}} Q^n_{\beta_0,\pi,\Sigma(F)}(A^{\delta_\epsilon} \backslash A \,\cap\, s's \leq \overline{e} \,\cap\, C^c(\underline{c}_1, \overline{c}_2^*)),$$

is smaller than or equal:

$$\sup_{(\pi,F)\in\Pi_2^n(c^*)\times\mathcal{F}} Q^n_{\beta_0,\pi,\Sigma(F)}(A^{\delta_\epsilon} \backslash A \,\cap\, s's \leq \overline{e} \,\cap\, t't \leq \underline{c}_1),$$

plus

$$\sup_{(\pi,F)\in\Pi_2^n(c^*)\times\mathcal{F}} Q^n_{\beta_0,\pi,\Sigma(F)}(A^{\delta_\epsilon} \backslash A \,\cap\, s's \leq \overline{e} \,\cap\, t't > \underline{c}_2^*),$$